An integer valued SU(3) Casson invariant for homology 3-spheres

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joint work with
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Notation

All 3-manifolds considered here are integral homology spheres ($\mathbb{Z}$HS) unless otherwise specified.

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Casson defined $\lambda_{SU(2)}(X)$ as a signed count of irreducible $SU(2)$ representations $\rho : \pi_1 X \to SU(2)$ up to conjugation.

Theorem [Taubes]

$$\lambda_{SU(2)}(X) = \sum_{[A] \in \mathcal{M}^*} (-1)^{SF(\theta, A)}.$$

Here, $\mathcal{M}$ is the SU(2) **Flat Moduli Space** and $\mathcal{M}^*$ the subspace of irreducible flat connections. It may be necessary to perturb the flatness equations to ensure that $\mathcal{M}^*$ is finite, but the RHS is independent of (generic) perturbation.

Spectral Flow

$SF(\theta, A) = SF(K_{A_t})$, where A_t is a path of connections from the trivial connection θ to and A and K_{A_t} is the **odd signature operator** acting on

$$\Omega^0(X; su(2)) \oplus \Omega^1(X; su(2)),$$

$$K_A : (\xi, a) \mapsto (d_A^* a, d_A \xi - *d_A a).$$

If A is flat, then $\ker K_A =$

$$H_A^0(X; su(2)) \oplus H_A^1(X; su(2)).$$

Main Problem

For $SU(n)$, one can define

$$\lambda'_{SU(n)}(X;h) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{SF(\theta,A)},$$

but in contrast to the $n=2$ case, $\lambda'_{SU(n)}(X;h)$ is not independent of perturbation h.

To extract a well-defined invariant of homology spheres, we need to define a counterterm involving only reducible connections that corrects for this dependence.

Any reducible flat SU(3) connection on a $\mathbb{Z}$HS is either trivial or has image in $SU(2) \times \{1\}$ (up to gauge).

The tangent space to $\mathcal{A}$ is $\Omega^1(X; su(3))$, with adjoint action of $SU(2) \times \{1\}$ on the coefficients

$$su(3) = (su(2) \oplus \mathbb{R}) \oplus \mathbb{C}^2,$$

(adjoint $\oplus$ trivial) $\oplus$ canonical.

Split the spectral flow accordingly and write SF_ν for $SF_{\mathbb{C}^2}$. The correction term has form

$$\sum_{[B] \in \mathcal{M}_h^{red}} (-1)^{SF(\theta,B)} \left(\frac{SF_\nu(B_0, B)}{2} \right),$$

for some fixed reducible B_0.

Natural choice: $B_0 = \theta$.

Gauge ambiguity: $SF_\nu(\theta, B)$ depends on the choice of gauge representative $SF_\nu(\theta, g \cdot B) = SF_\nu(\theta, B) + 4\,\mathrm{deg}(g)$.

To fix this problem, we use the Chern-Simons function $cs : \mathcal{A} \to \mathbb{R}$ to define a counter term, since

$$cs(g \cdot A) = cs(A) + \mathrm{deg}(g).$$

$$\lambda''(X; h) = \sum_{[B] \in \mathcal{M}_h^{red}} (-1)^{SF(\theta,B)} \left[\frac{SF_\nu(\theta, B) - 4cs(\widehat{B}) + 2}{2} \right]$$

Theorem [B-Herald, 1998]

$$\lambda_{SU(3)}(X) = \lambda'(X;h) + \lambda''(X;h)$$

is independent of generic small h and satisfies $\lambda_{SU(3)}(-X) = \lambda_{SU(3)}(X)$.

Theorem [BH, 1999]

$$\lambda_{SU(3)}(X\#Y) = \lambda_{SU(3)}(X)+\lambda_{SU(3)}(Y) + 4\lambda_{SU(2)}(X)\lambda_{SU(2)}(Y).$$

Corollary

$\lambda_{SU(3)} - 2\lambda^2_{SU(2)}$ is additive under connected sum.

Although $\lambda''_{SU(3)}$ is the natural generalization of Walker's correction term, it has several unpleasant features:

1. Chern-Simons values are not known to be rational.
2. $\lambda_{SU(3)}$ is <u>not</u> a finite type invariant.
3. Even when $\lambda'(X, h)$ is well-defined independent of h (e.g. when $\mathcal{M}^{red}$ is nondegenerate), computing $\lambda''(X, h)$ is fraught with technical difficulties.

Definition

$\mathcal{M}^{red}$ is *nondegenerate* if

$$H^1_B(X; \mathbb{C}^2) = 0 \text{ for all } [B] \in \mathcal{M}^{red},$$

Next, we will define an invariant

$$\tau_{SU(3)} : \{\mathbb{Z}\text{HS}\} \to \mathbb{Z}$$

that carries the same information as $\lambda_{SU(3)}$ modulo $SU(2)$ gauge theory.

The basic idea is that with a different choice of basepoint(s), one can define a correction term τ'' with no gauge ambiguity, so no counter term is needed.

This new invariant has the advantage of being more readily computable. For instance, whenever $\mathcal{M}^{red}$ is nondegenerate, τ'' vanishes and $\tau_{SU(3)} = \lambda'$.

An integer valued correction term

Let $\mathcal{G}_0 \subset \mathcal{G}$ be the identity component (degree zero gauge transformations).

$$\begin{array}{cl} \mathcal{A} & \\ \downarrow & \\ \mathcal{A}/\mathcal{G}_0 & \\ \downarrow & \mathbb{Z}-\text{cover} \\ \mathcal{A}/\mathcal{G} & \end{array}$$

The last quotient introduces π_1, and spectral flow is well-defined upstairs on $\mathcal{A}/\mathcal{G}_0$.

Let $\widetilde{\mathcal{M}}$ be the lift of $\mathcal{M}$. Then

(i) $cs(g \cdot A) = cs(A) + \deg(g)$, and

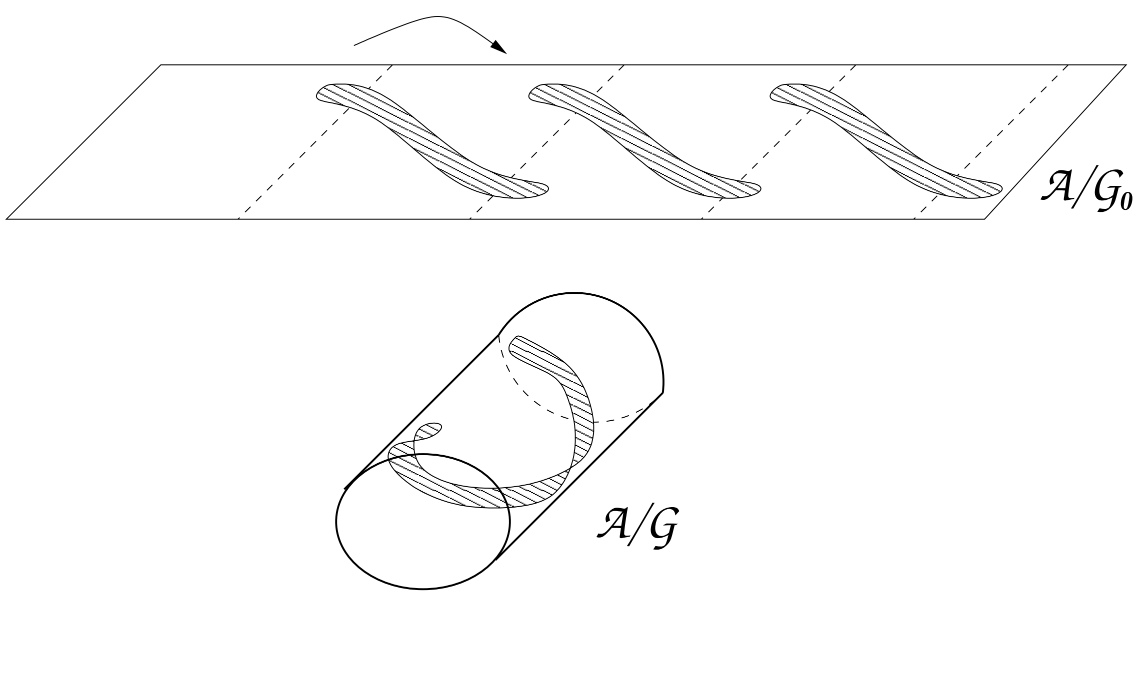
(ii) $cs : \mathcal{A} \to \mathbb{R}$ is constant on components of flat connections.

Lemma The cover $\widetilde{\mathcal{M}} \to \mathcal{M}$ is trivial.

Key Observation

Suppose $\mathcal{U} \subset \mathcal{A}/\mathcal{G}$ is connected and $\widetilde{\mathcal{U}} \to \mathcal{U}$ is the trivial $\mathbb{Z}$ cover.

In this situation, the spectral flow between any two points $[A], [B] \in \mathcal{U}$ can be defined unambiguously by lifting.



Number the components $C_1, \ldots, C_n$ of $\mathcal{M}^{red} = \mathcal{M}^*_{SU(2)}$. For each $i = 1, \ldots, n$, choose $[B_i^+], [B_i^-]$ on C_i with the property that $SF_\nu(B_i^-, B_i^+)$ is maximal.

Define $\tau''(X;h) =$

$$\frac{1}{4} \sum_{i=1}^{n} \sum_{\substack{[B] \in \mathcal{M}_h^{red} \\ \text{near } C_i}} (-1)^{SF(\theta,B)} \Big(SF_\nu(B_i^+, B) + SF_\nu(B_i^-, B) + \dim H^1_{B_i^-}(X; \mathbb{C}^2) \Big).$$

Claim $\tau''(X;h) \in \mathbb{Z}$.

Note that $B_i^\pm$ are $SU(2)$ connections. Identify $\mathbb{C}^2$ with $\mathbb{H}$ (quaternions) and the action of $SU(2) \subset \mathbb{H}$ with left multiplication.

Because $H^1_{B_i^-}(X;\mathbb{C}^2)$ is quaternionic, its dimension is a multiple of 4. Further, additivity of SF gives that

$$SF_\nu(B_i^+, B) + SF_\nu(B_i^-, B)$$
$$= 2SF_\nu(B_i^+, B) + SF_\nu(B_i^-, B_i^+).$$

Now $SF_\nu(B_i^+, B)$ is even because of the complex structure on the eigenspaces of K_{B_t}.

Also $SF_\nu(B_i^-, B_i^+)$ is divisible by 4 because the eigenspaces of K_{B_t} are quaternionic.

Theorem [B-H-Kirk]

Let $\tau_{SU(3)}(X) = \lambda'(X;h) + \tau''(X;h)$. Then $\tau_{SU(3)}(X) \in \mathbb{Z}$, is independent of h, and satisfies:

$$\tau_{SU(3)}(-X) = \tau_{SU(3)}(X).$$

$$\tau_{SU(3)}(X\#Y) = \tau_{SU(3)}(X) + \tau_{SU(3)}(Y) + 4\lambda_{SU(2)}(X)\lambda_{SU(2)}(Y).$$

Proof

Using what we already know about $\lambda_{SU(3)}$, it suffices to consider $\lambda_{SU(3)} - \tau_{SU(3)}$. For example, (ii) holds $\Leftrightarrow$ $\lambda_{SU(3)} - \tau_{SU(3)}$ is additive under connected sum.

We first describe $\lambda_{SU(3)} - \tau_{SU(3)}$ in terms of a more general construction.

Proposition

For any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \sum_{\substack{[B]\in\mathcal{M}_h^{red} \\ \text{near } C_i}} (-1)^{SF(\theta,B)} \alpha_i$$

is independent of generic small h.

If each component C_i is Bott-Morse, it contributes $\pm\chi(C_i)\alpha_i$ to this invariant (*weighted Casson* $SU(2)$ *invariant*).

Corollary

If the weights satisfy $\alpha_i(A\#\theta) = \alpha_i(A)$ and likewise for $\alpha_j(\theta\#B)$, then the resulting invariant is additive under connected sum.

Claim

$$\lambda_{SU(3)} - \tau_{SU(3)} = \lambda'' - \tau''$$
$$= \sum_{i=1}^{n} \sum_{\substack{[B]\in\mathcal{M}_h^{red} \\ \text{near } C_i}} (-1)^{SF(\theta,B)} \alpha_i$$

where

$$\alpha_i = \max_{[A]\in C_i} \left\{\rho(A) + \tfrac{1}{2}\dim H^1_A(X;\mathbb{C}^2)\right\}$$
$$+ \min_{[B]\in C_i} \left\{\rho(B) - \tfrac{1}{2}\dim H^1_B(X;\mathbb{C}^2)\right\}.$$

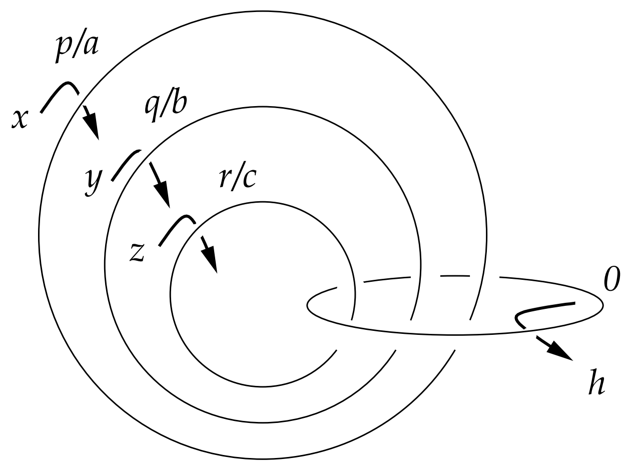
Here, $\rho(A)$ is the Atiyah-Patodi-Singer rho invariant associated to the flat $SU(2)$ connection via its fundamental representation on $\mathbb{C}^2$. I.e.

$$\rho(A) = SF(\theta, A) - 4cs(A) + 2 - \dim H^1_A(X;\mathbb{C}^2)$$

(see [Kirk Klassen Ruberman]).

Brieskorn spheres

Fix p, q, r pairwise relatively prime and consider the Brieskorn homology sphere $\Sigma(p, q, r)$.



Taking a, b, c with

$$aqr + bpr + cpq = 1,$$

then $\pi_1\Sigma$ has presentation

$$\langle x, y, z, h \mid h \text{ central}, x^p = h^a, y^q = h^b, \\ z^r = h^c, xyz = 1\rangle.$$

Proposition

If $\alpha : \pi_1\Sigma \to SU(3)$ is irreducible, then $\alpha(h) = e^{2\pi ik/3}I$ is central.

If $\alpha : \pi_1\Sigma \to SU(3)$ is reducible, then up to conjugation $im(\alpha) \subset SU(2) \times \{1\}$ and

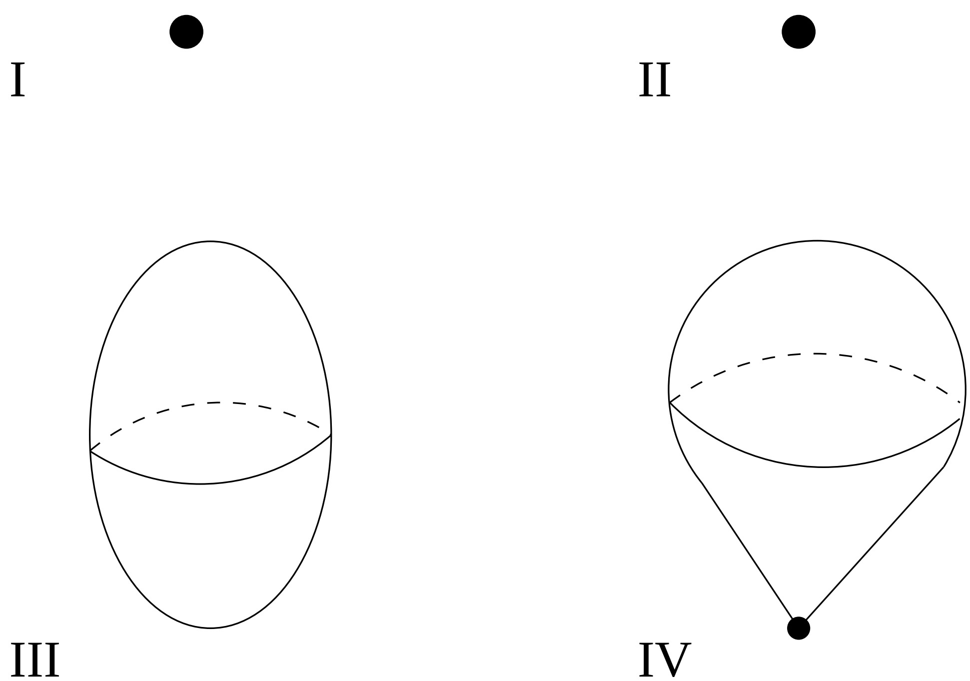
$$\alpha(h) = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Path components of

$$R(\Sigma, SU(3)) = \mathcal{M}_{SU(3)}(\Sigma)$$

are indexed by fixing $\alpha(h)$ and choosing $\alpha(x), \alpha(y)$ and $\alpha(z)$ to be p-th, q-th and r-th roots of h^a, h^b and h^c, respectively.

Using the Seifert fibration $\Sigma \to S^2$, one can interpret $R(\Sigma, SU(3))$ in terms of moduli spaces of parabolic bundles over S^2 with 3 marked points.



$R(\Sigma, SU(3))$ contains 4 types of components. Type II and IV contain reducibles, but only type IV components are singular.

In case $p = 2$, there are no 2-sphere components. In this case, $\tau_{SU(3)}(\Sigma)$ is simply a count of the Type I points.

$\mathbb{Z}$**HS**	$\tau_{SU(3)}(\Sigma)$
$\Sigma(2,3,6k \pm 1)$	$3k^2 \pm k$
$\Sigma(2,5,10k \pm 1)$	$33k^2 \pm 9k$
$\Sigma(2,5,10k \pm 3)$	$33k^2 \pm 19k + 2$
$\Sigma(2,7,14k \pm 1)$	$138k^2 \pm 26k$
$\Sigma(2,7,14k \pm 3)$	$138k^2 \pm 62k + 4$
$\Sigma(2,7,14k \pm 5)$	$138k^2 \pm 102k + 16$
$\Sigma(2,9,18k \pm 1)$	$390k^2 \pm 58k$
$\Sigma(2,9,18k \pm 5)$	$390k^2 \pm 210k + 24$
$\Sigma(2,9,18k \pm 7)$	$390k^2 \pm 298k + 52$

If $p > 2$, then $R(\Sigma, SU(3))$ always contains pointed 2-spheres. To resolve them, we use twisting perturbations.

Decomposing Σ along a torus

Let $Y = S^1 \times D^2$ be a neighborhood of the singular r-fiber in $\Sigma(p,q,r)$ and set $Z = \Sigma - Y$. Then

$$\Sigma = Y \cup_T Z.$$

We perturb the flatness equations in the solid torus Y and study the effect on a pointed 2-sphere.

Theorem [Floer]

If A is perturbed flat on Σ, then its restriction $A|_Z$ is flat.

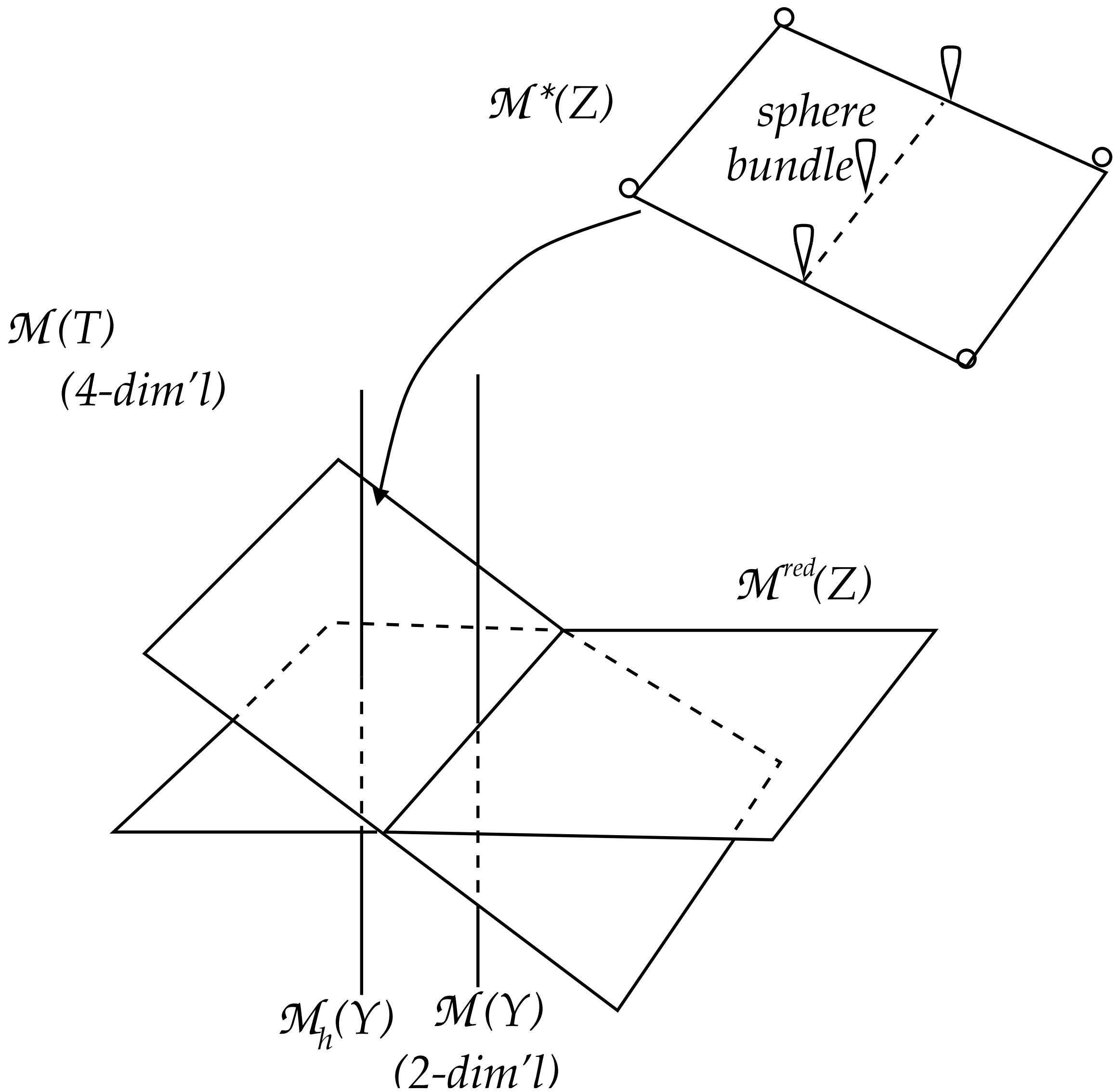
We view $\mathcal{M}(\Sigma)$ and $\mathcal{M}_h(\Sigma)$ as an intersection in the 4-dimensional "SU(3) Pillowcase"

$$\mathcal{M}(T) = T^2 \times T^2/S_3,$$

where S_3 acts diagonally.

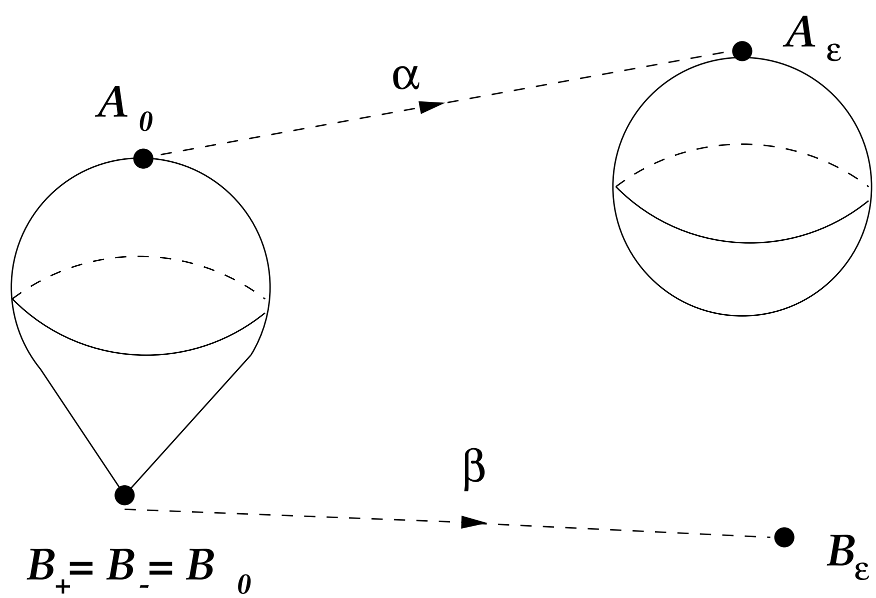
$$\begin{array}{ccccc} & & \mathcal{M}_h(Y) & & \\ & \nearrow & & \searrow & \\ \mathcal{M}_h(\Sigma) & & & & \mathcal{M}(T) \\ & \searrow & & \nearrow & \\ & & \mathcal{M}(Z) & & \end{array}$$

The restriction map $\nearrow$ collapses 2-spheres to points.



Perturbing moves this intersection point transverse to the arc of pointed 2-spheres. The restriction map $\mathcal{M}^*(Z) \to \mathcal{M}(T)$ is submersive at this new point and has nondegenerate 2-sphere fibers.

Thus $\mathcal{M}_h(\Sigma) = \mathcal{M}(Z) \cap \mathcal{M}_h(Y)$ consists of one isolated reducible orbit and a smooth nondegenerate 2-sphere of irreducible orbits.



The 2-sphere is Bott-Morse, and so contributes its Euler characteristic to the $SU(3)$ Casson invariant.

Spectral flow of reducibles

Let B_t be the path of h_t-perturbed flat reducible connections near a pointed 2-sphere. To calculate $SF_\nu(B_0, B_\varepsilon)$, we split the spectral flow according to the manifold decomposition

$$\Sigma = Y \cup_T Z.$$

To get a self-adjoint operator on manifolds with boundary, we use the APS boundary conditions $P+$ and P^- with respect to the tangential operator

$$S_a : \Omega^{0+1+2}(T; \mathbb{C}^2) \to \Omega^{0+1+2}(T; \mathbb{C}^2)$$

$$S_a(\alpha, \beta, \gamma) = (*d_a\beta, - * d_a\alpha - d_a * \gamma, d_a * \beta)\,.$$

Here, $a = A|_T$ is a connection on T and $P_a^\pm$ are the positive and negative eigenspans of S_a on L^2 forms.

Taking paths of reducible connections on Y or Z in cylindrical form, then both $SF_\nu(B_t, P^+; Y)$ and $SF_\nu(B_t, P^-; Z)$ become well-defined. If a_t is nontrivial, then $H^{0+1+2}_{a_t}(T; \mathbb{C}^2) = 0$.

Theorem [Nicolaescu]

$$SF_\nu(B_t; \Sigma) = SF_\nu(B_t, P^+; Y) + SF_\nu(B_t, P^-; Z).$$

Claim

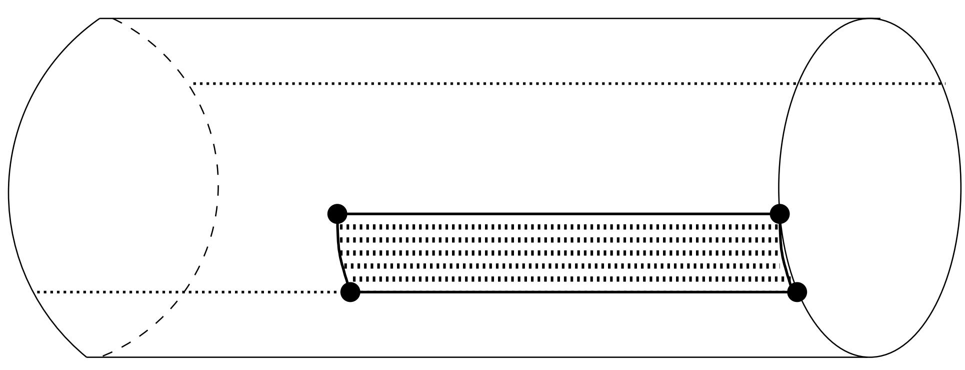
For the path B_t of reducible perturbed flat connections near a pointed 2-sphere,

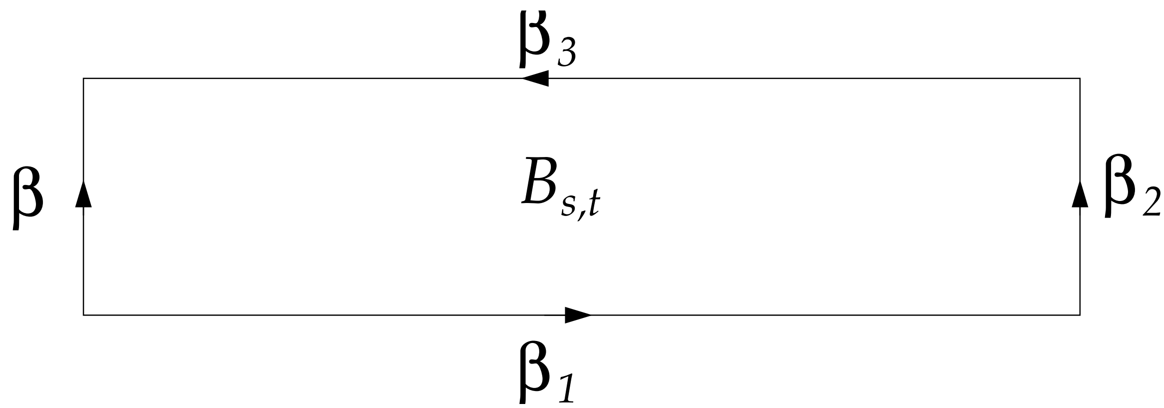
(i) $SF_\nu(B_t, P^+; Y) = 0$,

(ii) $SF_\nu(B_t, P^-; Z) = -2$.

Part (i) is not difficult; it follows from vanishing of cohomology $H^{0+1}_{B_0}(Y;\mathbb{C}^2)$. The argument for part (ii) uses additivity of the spectral flow under composition of paths.

Because $\mathcal{M}^{red}_Z$ is a union of open cylinders $S^1 \times \mathbb{R}$, there is a two parameter family $B_{s,t}$ of flat connections with $B_{0,t} = B_t$ and $B_{1,t}$ abelian.





The path β is homotopic to $\beta_1 \cdot \beta_2 \cdot \beta_3$, where dim $H^1_B(Z; \mathbb{C}^2)$ is constant along β_1 and β_3. Thus

$$SF(\beta) = SF(\beta_1 \cdot \beta_2 \cdot \beta_3) = SF(\beta_2).$$

Moreover, $\beta_2 \subset \mathcal{M}^{ab}(Z)$ is a path of abelians. Along β_2, at $t = 0$ there are four zero modes and for $t > 0$ there are none. As t increases from $t = 0$, two of the zero modes go up and the other two go down, hence

$$SF_\nu(B_t; Z; P^-) = SF(\beta_2) = -2.$$

Theorem

Each pointed 2-sphere contributes +2 to $\tau_{SU(3)}(\Sigma(p,q,r))$.

$\mathbb{Z}$**HS**	$\tau_{SU(3)}(\Sigma)$
$\Sigma(3,4,12k\pm1)$	$105k^2 \pm 21k$
$\Sigma(3,4,12k\pm5)$	$105k^2 \pm 87k + 16$
$\Sigma(3,5,15k\pm1)$	$276k^2 \pm 40k$
$\Sigma(3,5,15k\pm2)$	$276k^2 \pm 74k + 2$
$\Sigma(3,5,15k\pm4)$	$276k^2 \pm 148k + 16$
$\Sigma(3,5,15k\pm7)$	$276k^2 \pm 254k + 56$

Let $K_{p,q}$ = the (p,q) torus knot and set $X_n = 1/n$ Dehn surgery on $K_{p,q}$. Then $X_n = \pm\Sigma(p,q,r)$ for $r = |pqn-1|$.

(p,q)	$\tau_{SU(3)}(X_n)$
(2,3)	$3n^2 - n$
(2,5)	$33n^2 - 9n$
(2,7)	$138n^2 - 26n$
(2,9)	$390n^2 - 58n$
(2,11)	$885n^2 - 107n$
(2,13)	$1743n^2 - 179n$
(2,15)	$3108n^2 - 276n$
(3,4)	$105n^2 - 21n$
(3,5)	$276n^2 - 40n$
(3,7)	$1128n^2 - 124n$
(3,8)	$1953n^2 - 179n$
(3,10)	$4851n^2 - 367n$
(3,11)	$7140n^2 - 476n$
(3,13)	$14028n^2 - 812n$
(3,14)	$18915n^2 - 993n$
(4,5)	$1011n^2 - 111n$
(4,7)	$4110n^2 - 320n$
(4,9)	$11490n^2 - 712n$
(4,11)	$25935n^2 - 1297n$
(4,13)	$50925n^2 - 2171n$
(4,15)	$90636n^2 - 3320n$
(4,17)	$149940n^2 - 4888n$

Conjectures

1. $\tau_{SU(3)} : \{\mathbb{Z}\text{HS}\} \to 2\mathbb{Z}$ (proved 2002 BHK).

2. For any knot K in S^3,

$$\tau_{SU(3)}(X_n) = A(K)n^2 + O(n)$$

has quadratic growth in n with

$$A(K) \ = \ 2c_4(K) + c_2(K)^2$$

where $\sum_{i \geq 0} c_{2i}(K)z^{2i}$ is the Conway polynomial of K.

3. For surgeries on $(2,q)$ torus knots,

$$\tau_{SU(3)}(X_n) = A(K)n^2 + B(K)n,$$

where

$$A(K_{2,q}) = \frac{(q^2-1)(q^2-3)}{16}$$

$$B(K_{2,q}) = \frac{(q \pm 1)(q^2 \mp q - 3)}{12}$$

with signs in B taken according to the value of q mod 4.

Questions

4. Is $\tau_{SU(3)}$ a finite type invariant? (Not order 6.)

5. What is the mysterious linear coefficient $B(K)$?