Problem in Higgs bdls:

Simpson $\Gamma \subset G$ lattice (say $\mathrm{rank}_{\mathbb{R}} G \geq 2$), $\Gamma = \pi_1$(cpct Kähler)

$\Rightarrow G$ of Hodge type (Cartan invol in conn comp of id)

Problem: G Hodge type, G/K not Herm sym

e.g. $SO(2p, q)$ $p > 1, q >$

is $\Gamma = \pi_1$(cpct Kähler) ??

Conj: No

many examples of non-unifm lattices $\Gamma \subset SO(2p, q)$ that are homo images of π_1(cpct Kähler).

Joint with Allcock, Carlson

$\mathcal{C}$ = (proj) space of cubic forms in $\mathbb{C}^4$

$\mathcal{C}_{st}$ ← stable: only nodes

$\mathcal{C}_{sm}$ ← smooth

$h = |z_0|^2 + |z_1|^2 + \cdots + |z_4|^2$

$\Gamma = PU(1, 4, \mathcal{E})$

$\mathcal{E} = \mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$ $\mathcal{E}^{4,1}$

$$\mathcal{C}_{sm} \quad \tilde{\mathcal{C}}_{sm} \xrightarrow{per} \Gamma\backslash B^4$$

Γ gen by cx refl of order 6 about $v^\perp$, $h(v) = 1$

$\mathcal{H} = \bigcup v^\perp$

Thm: $M_{st} = \mathcal{C}_{st}/G \xrightarrow{per} \Gamma\backslash B^4$ iso of analytic spaces

$M_{sm} = \mathcal{C}_{sm}/G \longrightarrow \Gamma\backslash(B^4 - \mathcal{H})$ iso of cx analytic orbifolds

$$\to \pi_2^{orb}\left(\Gamma\backslash(B^4-\mathcal{H})\right) \to \pi_1(G) \to \pi_1(\mathcal{C}_{sm}) \to \pi_1^{orb}\left(\frac{B^4-\mathcal{H}}{\Gamma}\right) \to 1$$

($\pi_2^{orb} = 0$)

Determines $\pi_1(\mathcal{C}_{sm})$ in terms of $\pi_1^{orb}\left(\Gamma\backslash(B^4-\mathcal{H})\right)$

and $1 \to \pi_1(B^4 - \mathcal{H}) \to \pi_1^{orb}\left(\frac{B^4-\mathcal{H}}{\Gamma}\right) \to \Gamma \to 1$

$\uparrow$ not finitely generated.

Fact: $\pi_1(\mathcal{C}_{sm})$ not a lattice in a Lie gp.

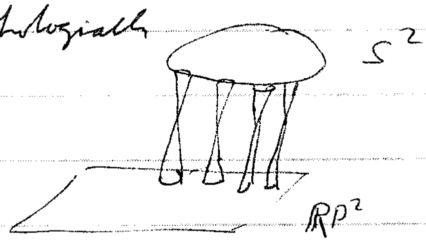
$$\begin{array}{ccc} \mathcal{C}^{\mathbb{R}} & \subset & \mathcal{C} \\ \cup & & \cup \\ \mathcal{C}^{\mathbb{R}}_{st} & \subset & \mathcal{C}_{st} \\ \cup & & \cup \\ \mathcal{C}^{\mathbb{R}}_{sm} & \subset & \mathcal{C}^{\mathbb{R}}_{sm} \end{array} \qquad \begin{array}{ccc} M^{\mathbb{R}}_{st} & \to & M_{st} \\ \mathcal{C}^{\mathbb{R}}_{st}/G_{\mathbb{R}} & \to & \mathcal{C}_{st}/G \\ \cup & & \cup \\ \mathcal{C}^{\mathbb{R}}_{sm}/G_{\mathbb{R}} & \to & \mathcal{C}_{sm}/G \\ M^{\mathbb{R}}_{sm} & \to & M_{sm} \end{array}$$

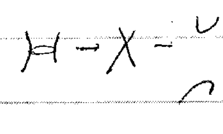
finite to one and surjective

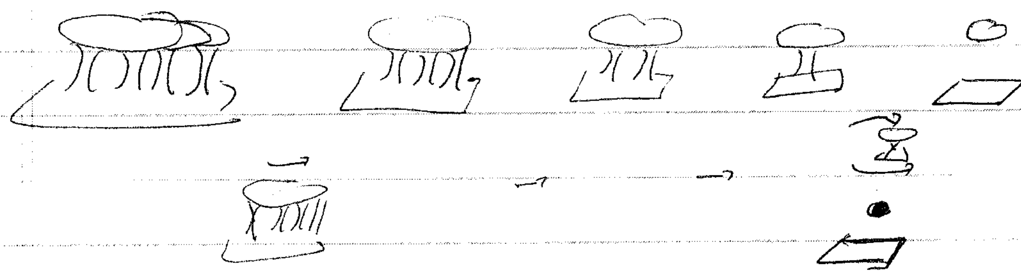
classical: Schläfli, Klein, Segre: $M^{\mathbb{R}}_{sm}$ has 5 connected comps

Klein's explanation: $\sum_{i=0}^{4} x_0 \cdots \hat{x}_i \cdots x_3 = 0$ 4-nodal surface

Topologically







real struc $\to$ ex conj $\to$ anti-linear invol of $\mathcal{E}^{4,1}$

[illegible] $\{ \quad (\longrightarrow \{ \quad)$

Thm: There are (up to $\pm$) 5 equiv classes of anti-hol invols of $\mathcal{E}^{4,1}$

$\tau_0 : (z_0, \ldots, z_4) \to (\bar z_0, \ldots, \bar z_4)$

$\tau_1 \quad (z_0, \ldots, z_n) \to (\bar z_0, \ldots \bar z_3, -\bar z_4)$

$\tau_2 \qquad\qquad (\bar z_0, \quad -\bar z_3, -\bar z_4)$

$\vdots$

$\tau_4 \qquad\qquad \to (\bar z_0, -\bar z_1, \ldots, -\bar z_4)$

Let $H_i = \mathrm{Fix}(\tau_i) \subset B^4$ a real hyp space $\qquad \mathrm{Fix}(\tau_i) \subset \mathcal{E}^{4,1}$

$\Gamma_i = Z(\tau_i) \subset \Gamma \qquad\qquad \underline{\mathbb{Z}^{1,4-i} \oplus \sqrt{3}\,\mathbb{Z}^i}$

$$M^{\mathbb{R}}_{sm} = \bigcup \Gamma_i \backslash (H_i - \mathcal{H}) = \bigcup M^{\mathbb{R}}_{sm,i}$$

<u>Thm</u> $\Gamma_i = PO(q_i, \mathbb{Z})$ where
$$q_0 = -x_0^2 + x_1^2 + \cdots + x_4^2$$
$$q_1 = -x_0^2 + \cdots + x_3^2 + 3x_4^2$$
$$\vdots$$
$$q_4 = -x_0^2 + 3x_1^2 + \cdots + 3x_4^2$$

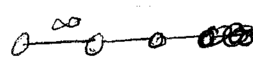
Thm: $\Gamma_0, \Gamma_3, \Gamma_4$ are Coxeter gps (gen by reflections)

Γ_1, Γ_2 have Cox subgps of index 2

diagrams as shown

Can compute $\pi_1^{orb}(M^{\mathbb{R}}_{sm,i})$:

$$\pi_1(M^{\mathbb{R}}_{sm,0}) = S_5$$
$$\pi_1(M^{\mathbb{R}}_{sm,1}) = (S_3 \times S_3) \rtimes \mathbb{Z}/2$$
$$\pi_1(M^{\mathbb{R}}_{sm,2}) = (D_\infty \times D_\infty) \rtimes \mathbb{Z}/2$$

$\pi_1(M^{\mathbb{R}}_{sm,3}) = \pi_1(M^{\mathbb{R}}_{sm,4}) =$  surjects to non-unif lattice in $O(3,1)$

$s_i s_j s_i = s_j s_i s_j$

$(s_i s_j)(s_i s_j)(s_i s_j) = 1$

all aspherical orbifolds: orb univ cover is contractible

$$\begin{tikzcd}
\pi_2^{orb}(M^R_{sm,c}) \arrow[r] \arrow[d, equal] & \pi_1(G^R) \arrow[r] \arrow[d, equal] & \pi_1(\mathcal{C}^R_{sm,c}) \arrow[r] & \pi_1^{orb}(M^R_{sm,c}) \arrow[r] & \pi_0(G^R) \arrow[r] \arrow[d, equal] & 1 \\
0 & \mathbb{Z}/2 \times \mathbb{Z}/2 & & & \mathbb{Z}/2 &
\end{tikzcd}$$

ex: $1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \pi_1(\mathcal{C}^R_{sm,0}) \to A_5 \to 1$

Check against Segre lists of monodromy grps

$$\pi_1(\mathcal{C}^R_{sm,c}) \to \mathrm{Aut}(\text{Conf of lines})$$

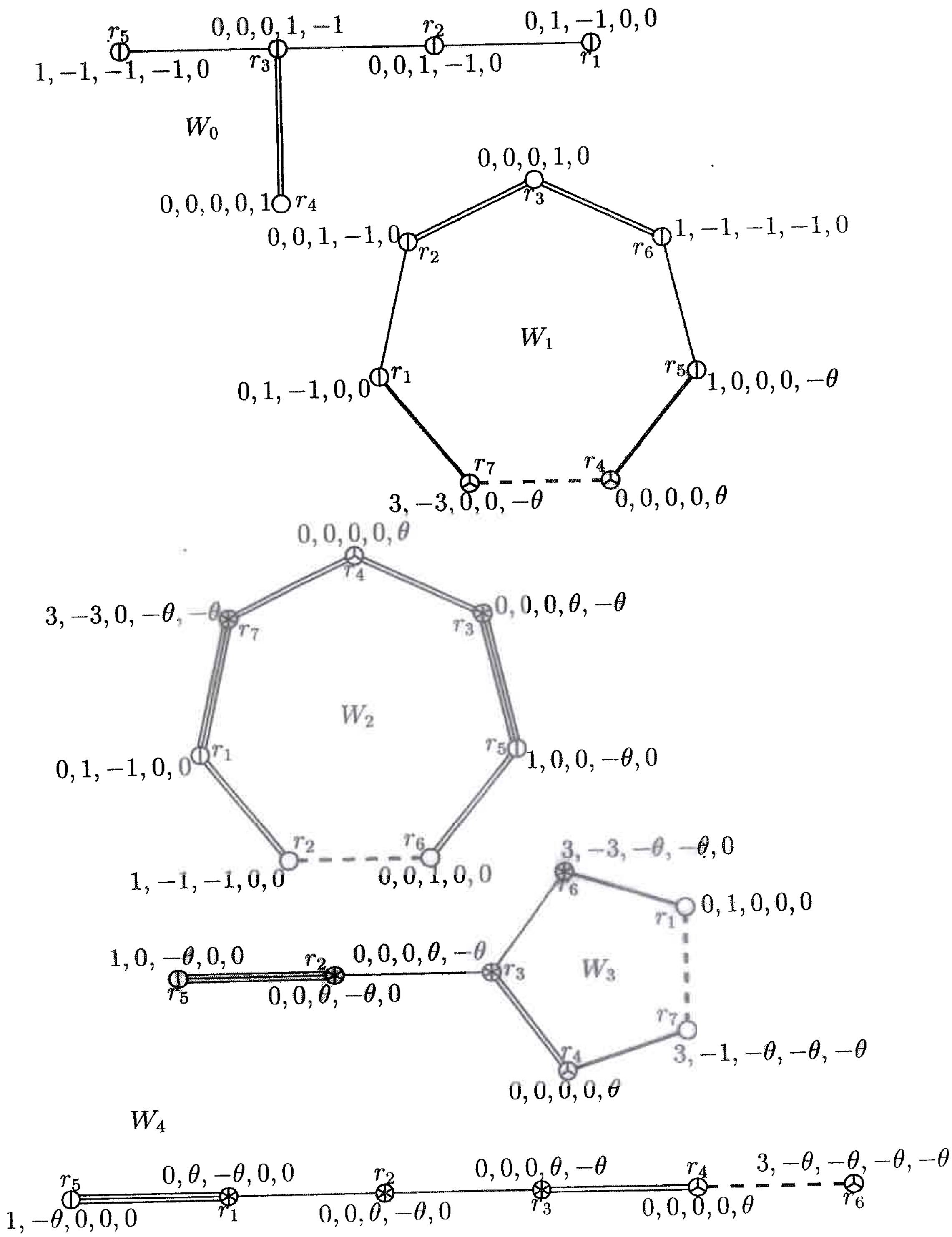


FIGURE 6.2. Simple roots for the W_j.