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Iterated Integrals and Motives

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1. Iterated Line Integrals

Kuo-Tsai Chen: For X a smooth manifold, $\gamma : [0, 1] \to X$ a piecewise smooth path, and $w_1, \ldots, w_r$ smooth 1-forms on X. Define

$$\int_\gamma w_1 \ldots w_r$$
$$:= \int_{0 \le t_1 \le t_2 \le \cdots \le t_n \le 1} f_1(t_1) \ldots f_r(t_r) dt_1 \ldots dt_r,$$

where $\gamma^* w_j = f_j(t) dt$.

Linear combinations of these are called *iterated line integrals*. They are considered as functions on PX, where

$$PX = \{\text{piecewise smooth } \gamma : [0, 1] \to X\}$$

An iterated integral line integral whose value is constant on homotopy classes of paths (rel endpoints) is said to be *closed*. Closed iterated line integrals give rise to multi-valued functions.

2. Examples

Polylogarithms: The k-logarithm

$$\mathrm{ln}_k(x) := \sum_{n \geq 1} \frac{x^n}{n^k} \qquad |x| < 1$$

can be expressed as the length k iterated integral

$$\int_0^x \frac{dz}{1-z} \overbrace{\frac{dz}{z} \cdots \frac{dz}{z}}^{k-1}$$

on $\mathbb{P}^1 - \{0, 1, \infty\}$. The value of this at $x = 1$ is $\zeta(k)$.

Multiple polylogarithms: Multiple polylogs, such as

$$L_{m_1,m_2}(x_1, x_2) := \sum_{0<k_1<k_n} \frac{x_1^{k_1} x_2^{k_2}}{k_1^{m_1} k_2^{m_2}} \qquad |x_j| < 1$$

are iterated integrals.

For example,

$$L_{1,1}(x,y) = \int_{(0,0)}^{(x,y)} \left(\frac{dy}{1-y} \frac{dx}{1-x} + \frac{d(xy)}{1-xy} \left(\frac{dy}{1-y} - \frac{dx}{1-x} - \frac{dx}{x} \right) \right).$$

on

$$\mathbb{C}^2 - \{(x,y) : xy(1-x)(1-y)(1-xy) \neq 0\}$$

Theta functions: On the torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\operatorname{Im}\tau > 0$, we have

$$\log\theta(x) = \log\theta(x_o) + \frac{\pi}{\operatorname{Im}\tau} \left(\int_{x_o}^{x} (d\overline{z}\, dz + \xi) - \frac{1}{2}\Big(z(x) - \overline{z}(x_o)\Big)^2 + \frac{1}{2}\Big(z(x_o) - \overline{z}(x_o)\Big)^2 \right),$$

where $z(x) = \int_0^x dz$, which is the elliptic logarithm.

3. Iterated Integrals of Higher Forms (Chen)

Suppose that $w_1, \ldots, w_r$ are differential forms on X, all of positive degree. We have

$$\int w_1 w_2 \ldots w_r \in E^{-r+\sum_j \deg w_j}(PX).$$

Up to a sign

$$\int w_1 w_2 \ldots w_r = \pi_* \phi^* (p_1^* w_1 \wedge p_2^* w_2 \wedge \cdots \wedge p_r^* w_r)$$

where

1. $\Delta^r = \{(t_1, \ldots, t_r) : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1\}$ is the *time ordered* form of the standard r-simplex,

2. $\phi : \Delta^r \times PX \to X^r$ is the *sampling map*

$$\phi(t_1, \ldots, t_r, \gamma) = \Big(\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_r)\Big),$$

3. π_* denotes integration over the fiber of the projection

$$\pi : \Delta^r \times PX \to PX.$$

4. Properties

For natural subsets Z of PX, such as the space of paths $P_{x,y}X$ from x to y, we let $Ch^\bullet(Z)$ be the subcomplex of $E^\bullet(Z)$ generated by iterated integrals. The basic properties of iterated integrals can be summarized by:

- $Ch^\bullet(PX)$ is a commutative dg algebra;
- $Ch^\bullet(P_{x,x}X)$ is a dg Hopf algebra with antipode;
- The complexes $Ch^\bullet(PX)$, $Ch^\bullet(P_{x,x}X)$, etc, can all be described algebraically in terms of various flavors of the (reduced) bar construction. For example:

  $$Ch^\bullet(P_{x,y}X) = B(\mathbb{R}, E^\bullet(X), \mathbb{R})$$

  with the $E^\bullet(X)$ actions on $\mathbb{R}$ given by the augmentations $E^\bullet(X) \to \mathbb{R}$ associated to x and y.

For example, in $Ch^\bullet(P_{x,x}X)$, we have:

Differential:

$$d\int w_1 \dots w_r = \sum_{j=1}^{r} \pm \int w_1 \dots dw_j \dots w_r$$
$$+ \sum_{j=1}^{r-1} \pm \int w_1 \dots w_{j-1}(w_j \wedge w_{j+1}) w_{j+2} \dots w_r$$

Shuffle product:

$$\int w_1 \dots w_r \wedge \int w_{r+1} \dots w_{r+s}$$
$$= \sum_{\sigma \in \mathrm{sh}(r,s)} \pm \int w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(r+s)}$$

Coproduct (induced by $(\alpha, \beta) \mapsto \alpha\beta$):

$$\int w_1 \dots w_r \mapsto \sum_{j=1}^{r} \int w_1 \dots w_j \otimes \int w_{j+1} \dots w_r$$

Antipode (induced by $\gamma \mapsto \gamma^{-1}$):

$$\int w_1 \dots w_r \mapsto \pm \int w_r \dots w_1$$

5. Chen's de Rham Theorems

Theorem 1 *If X is a simply connected manifold, then integration induces a natural Hopf algebra isomorphism*

$$H^\bullet(Ch^\bullet(P_{x,x}X)) \cong H^\bullet(P_{x,x}X;\mathbb{R}).$$

Corollary 2 *If X is simply connected, integration induces an isomorphism*

$$H^+(Ch^\bullet(P_{x,x}X))/H^+(Ch^\bullet(P_{x,x}X))^2 \cong \mathrm{Hom}(\pi_\bullet(X,x),\mathbb{R})[1].$$

Theorem 3 *The integration pairing*

$$H^0(Ch^\bullet(P_{x,x}X)) \otimes \mathbb{Z}\pi_1(X,x) \to \mathbb{C}$$

is a pairing of Hopf algebras under which

$$H^0(Ch_s^\bullet(P_{x,x}X))$$

annihilates J^{s+1}. The induced mapping

$$H^0(Ch_s^\bullet(P_{x,x}X)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X,x)/J^{s+1},\mathbb{C})$$

is an isomorphism.

6. What iterated integrals do not compute

When X is not simply connected, $H^\bullet(Ch^\bullet(P_{x,x}X))$ may have very little to do with $H^\bullet(P_{x,x}X)$ and $\pi_\bullet(X,x)$. This is because a mapping $f : X \to Y$ which induces an isomorphism on real cohomology, induces an isomorphism

$$H^\bullet(Ch^\bullet(P_{f(x),f(x)}Y)) \to H^\bullet(Ch^\bullet(P_{x,x}X)).$$

For example, suppose that $X = B\Gamma$, where Γ is perfect. Since the mapping $B\Gamma \to B\Gamma^+$ is a homology isomorphism

$$H^\bullet(Ch^\bullet(P_{f(x),f(x)}B\Gamma^+)) \to H^\bullet(Ch^\bullet(P_{x,x}B\Gamma))$$

is an isomorphism.

One of my main points will be that, in the theory of motives, $H^\bullet(Ch^\bullet(P_{x,x}X))$ may be a more interesting and more fundamental invariant of (X,x) than $H^\bullet(P_{x,x}X)$ or $\pi_\bullet(X,x)$.

7. Hodge Theory

There are natural Hodge and weight filtrations on $Ch^\bullet(P_{x,y}X)$ which make it into a mixed Hodge complex (MHC). When X is a compact Kähler manifold, these filtrations are defined as follows:

Hodge filtration: $F^p Ch^\bullet(P_{x,y}X)$ is the span of

$$\int w_1 \dots w_r$$

where $w_j \in F^{p_j} E^\bullet(X)$, where $p_1 + \cdots + p_r \geq p$.

Weight filtration: This is simply the filtration by length:

$$\begin{aligned} W_m Ch^\bullet(P_{x,y}X) &= Ch^\bullet_m(P_{x,y}X) \\ &:= \{\text{iterated integrals of length } \leq m\}. \end{aligned}$$

When X is the complement of a normal crossings divisor, the weight filtration also takes into account the number log singularities.

These induces a MHS on $H^\bullet(Ch^\bullet(P_{x,y}X))$.

8. Extensions of MHS

If H is a Hodge structure of negative weight,

$$\mathrm{Ext}^1_{\mathrm{Hodge}}(\mathbb{Z}, H) = H_{\mathbb{C}}/(H_{\mathbb{Z}} + F^0 H)$$

This is a complex torus which I'll denote by JH. It is compact when H has weight -1. If X is smooth projective, then $J(H_{2d+1}(X)(-d))$ is a Griffiths intermediate jacobian of X.

Extensions from cycles:

A homologically trivial algebraic d-cycle in X gives rise to an element of $J(H_{2d+1}(X)(-d))$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{2d+1}(X) & \longrightarrow & H_{2d+1}(X,Z) & \longrightarrow & H_{2d}(Z) & \longrightarrow & H_{2d}(X) \\
 & & \| & & \uparrow & & \uparrow \scriptstyle{\mathrm{class}_Z} & & \\
0 & \longrightarrow & H_{2d+1}(X) & \longrightarrow & E_Z & \longrightarrow & \mathbb{Z}(d) & \longrightarrow & 0
\end{array}$$

Extensions from iterated integrals:

If $H^1(X)$ vanishes, one has the exact sequence of $\mathbb{Z}$-MHS

$$0 \to H^3(X) \to H^2(ICh_2^\bullet(P_{x,x}X)) \\ \to H^2(X)^{\otimes 2} \xrightarrow{\text{cup}} H^4(X)$$

This gives rise to the element

$$0 \to H^3(X) \to E \to K \to 0 \qquad (*)$$

of

$$\text{Ext}^1_{\text{Hodge}}(K, H^3(X)) = J\,\text{Hom}(K, H^3(X))$$

which is independent of $x \in X$, where K is the kernel of the cup product $H^2(X)^{\otimes 2} \to H^4(X)$.

9. The Carlson-Clemens-Morgan Theorem

Denote the class of a divisor D in the Neron-Severi group

$$NS(X) := \{\text{divisors in } X\}/(\text{hom. equiv.})$$

of X by $[D]$. If the codimension 2 cycle

$$Z := \sum_{j,k} n_{jk}\, D_j \cap D_k$$

is homologically trivial, where the n_{jk} are integers (symmetric in jk) and the D_j divisors, then

$$\widehat{Z} := \sum_{j,k} n_{jk}\,[D_j] \otimes [D_k] \in H^2(X;\mathbb{Z})^{\otimes 2}$$

is an integral Hodge class of type $(2,2)$ in K. Pulling back the extension (*) along the mapping $\mathbb{Z}(-2) \to K$ that takes 1 to $\widehat{Z}$, we obtain the extension

$$0 \to H^3(X;\mathbb{Z}(2)) \to E_Z \to \mathbb{Z} \to 0$$

which determines a point

$$\phi_Z \in J(H^3(X)(2)).$$

On the other hand, the homologically trivial cycle Z determines a point

$$\Phi(Z) \in J(H_{2d-3}(X)(d-2)) = J(H^3(X)(2))$$

Theorem 4 (Carlson-Clemens-Morgan) *The points ϕ_Z and Φ_Z of $J(H^3(X)(2))$ are equal.*

Remark: Carlson-Clemens-Morgan phrased their result in terms of the MHS on $\pi_3(X,x)^*$ rather than that on $H^2(ICh_2^\bullet(P_{x,x}X))$. But for this they needed to assume that X is simply connected, in which case these two MHSs determine each other. For the formulation above, one only need assume that $H^1(X)$ vanishes. A more general version, and a heuristic proof, is given in math.AG/0109204.

10. Green's Interpretation

Denote the Chow group of codimension a algebraic cycles in X mod rational equivalence by $CH^a(X)$. If one wants to understand the product

$$CH^a(X) \otimes CH^b(X) \to CH^{a+b}(X)$$

the first thing one may look at is:

$$CH^a(X) \otimes CH^b(X) \to \Gamma H^{2a+2b}(X; \mathbb{Z}(a+b))$$

After this, one may consider the "crossover mapping":

$$\ker\left\{CH^a(X) \otimes CH^b(X) \to \Gamma H^{2a+2b}(X; \mathbb{Z}(a+b))\right\} \to \mathrm{Ext}^1_{\mathrm{Hodge}}(\mathbb{Z}, H^{2a+2b-1}(X; \mathbb{Z}(a+b))).$$

What Green observed is that when X is a simply connected projective manifold and $a = b = 1$, the result of Carlson-Clemens-Morgan implies this mapping is determined by the class

$$e(X) \in \mathrm{Ext}^1_{\mathrm{Hodge}}(K, H^3(X; \mathbb{Z}(2)))$$

discussed above. There is a natural mapping

$$\ker\left\{CH^1(X) \otimes CH^1(X) \to \Gamma H^4(X, \mathbb{Z}(2))\right\} \to \Gamma K(2).$$

The result of Carlson-Clemens-Morgan implies that cupping this homomorphism with $e(X)$ gives the "crossover mapping." He suggested that the ring structure of $CH^\bullet(X)$ (or its associated graded) is determined by the MHS on $\pi_\bullet(X, x)$.

For more details on Green's ideas and some partial results, see Archava's UCLA thesis.

11. Motivic Significance

This result (and its heuristic generalization) suggests that iterated integrals have a deep motivic significance. The discussion also suggests that it is $H^\bullet(Ch^\bullet(P_{x,x}X))$ that is motivically significant, rather then $H^\bullet(P_{x,x}X)$ or $\pi_\bullet(X,x)$.

Other evidence that $H^\bullet(Ch^\bullet(P_{x,x}X))$ has a motivic significance includes:

- Polylogs — particularly Terasoma's work on mixed zeta numbers (Zagier's conjecture);
- Bruno Harris's work on "harmonic volume" — the story of the algebraic 1-cycle $C - C^-$ in Jac C (periods of this cycle determined by periods of the fundamental group of C;

- The algebraic version of iterated integrals pioneered by Wojtkowiak, which includes algebraic de Rham theorems;

- The crystalline versions due to Shiho and Kim-Hain;

- The fact that the $\mathbb{Q}_\ell$ version has a natural Galois action for varieties defined over a number field.

- The compatibility of these Hodge, algebraic de Rham, Galois, and crystalline structures on $H^\bullet(Ch^\bullet(P_{x,y}X))$ with respect to the natural comparison isomorphisms.

12. Adams' Cobar Construction

Iterated integrals are the de Rham realization of the cosimplicial version of Adams' cobar construction. The cobar construction dates from 1953 and has roots in a slightly earlier paper of Adams and Hilton. The cosimplicial version appears to be due to Bousfield and Kan in the 1970s.

The original cobar construction gives a functor

$$\text{cobar} : \{\text{pointed topological spaces}\} \to \{\text{differential graded algebras}\}$$

and a chain map

$$\text{cobar}(X,x) \to \{\text{cubical chains on } P_{x,x}X\}$$

such that there are algebra isomorphisms

$$H_0(\text{cobar}(X,x)) \cong \mathbb{Z}\pi_1(X,x)$$

in general, and, when X is simply connected,

$$H_\bullet(\text{cobar}(X,x)) \cong H_\bullet(P_{x,x}X;\mathbb{Z}).$$

I'll explain the cosimplicial version first.

13. simplicial and Cosimplicial Spaces

The category of finite ordinals $\boldsymbol{\Delta}$ — its objects are the finite ordinals $[n] := \{0, 1, \ldots, n\}$; its morphisms are order preserving functions.

The *face map*

$$d^j : [n-1] \to [n], \quad 0 \leq j \leq n$$

is the unique order preserving injection that omits the value j.

A contravariant functor $\boldsymbol{\Delta} \to \mathcal{C}$ is called a *simplicial object* in the category $\mathcal{C}$. A *cosimplicial object* of $\mathcal{C}$ is a covariant functor $\boldsymbol{\Delta} \to \mathcal{C}$.

Example of cosimplicial space: Denote the standard n-simplex by Δ^n. We can regard its vertices as being the ordinal $[n]$. Each order preserving mapping $f : [n] \to [m]$ induces a linear mapping $|f| : \Delta^n \to \Delta^m$. These assemble to give the cosimplicial space $\Delta^\bullet$

$$\Delta^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \Delta^1 \underset{d^2}{\overset{d^0}{\rightrightarrows}} \Delta^2 \underset{d^3}{\overset{d^0}{\rightrightarrows}} \Delta^3 \quad \cdots$$

whose value on $[n]$ is Δ^n.

Example of a simplicial space: Suppose that K is an ordered finite simplicial complex. Then one has the simplicial set $K_\bullet$ whose set of n-simplices K_n is the set of order preserving mappings $\phi : [n] \to K$ (not necessarily injective) such that the images of the $\phi(j)$ span a simplex of K.

In particular, one has $I_\bullet$, the simplicial model of the unit interval:

$$I_n = \Big\{\text{order preserving mappings } [n] \to \{0,1\}\Big\}.$$

14. The Cosimplicial Model of PX

The cosimplicial model of the path space of a topological space X is

$$X^{I_\bullet} = \text{Hom}(I_\bullet, X).$$

Its space of n-cosimplices is $\text{Hom}(I_n, X)$. Since there are $n+2$ order preserving mappings $[n] \to \{0,1\}$, this is just X^{n+2}. The jth coface mapping $d^j : X^{I_{n-1}} \to X^{I_n}$ is

$$\overbrace{\text{id} \times \cdots \times \text{id}}^{j} \times (\text{diag}) \times \overbrace{\text{id} \times \cdots \times \text{id}}^{n-j} : X^{n+1} \to X^{n+2}$$

Denote it by $P^\bullet X$ and its set of n-cosimplices by $P^n X$.

15. The Cosimplicial Model of $P_{x,y}X$

Since $(\partial I)_n$ consists of the two constant maps $[n] \to \{0, 1\}$, the cosimplicial space $X^{\partial I_\bullet}$ consists of $X \times X$ in each degree. The mapping $X^{I_\bullet} \to X^{\partial I_\bullet}$ corresponds to the projection $PX \to X \times X$ that takes a path γ to its endpoints.

One obtains a cosimplicial model $P^\bullet_{x,y}X$ for $P_{x,y}X$ by taking the fiber of $X^{I_\bullet} \to X^{\partial I_\bullet}$ over (x, y). Note that $P^n_{x,y}X = X^n$.

16. Geometric Realization

Just as each simplicial space has a geometric realization, a cosimplicial space $X[\bullet]$ has a kind of geometric realization $\left\| X[\bullet] \right\|$, called the *total space associated to* $X^\bullet$. It is the subspace of

$$\prod_{n \geq 0} X[n]^{\Delta_n}$$

consisting of all sequences compatible with all morphisms $f : [n] \to [m]$ in $\mathbf{\Delta}$, where $X[n]^{\Delta_n}$ denotes the set of continuous mappings from Δ_n to $X[n]$ endowed with the compact-open topology.

Continuous mappings from a topological space Z to $\left\| X[\bullet] \right\|$ correspond naturally to continuous mappings

$$\Delta^\bullet \times Z \to X[\bullet]$$

of cosimplicial spaces.

17. Comparison with Path Spaces

Regard Δ^n as the time ordered simplex

$$\Delta^n = \{(t_1, \ldots, t_n) : 0 \le t_1 \le \cdots \le t_n \le 1\}.$$

There are continuous mappings

$$PX \to \|P^\bullet X\| \text{ and } P_{x,y}X \to \|P^\bullet_{x,y}X\|$$

given by the sampling map

$$\gamma \mapsto \{(t_1, \ldots, t_n) \mapsto (\gamma(0), \gamma(t_1), \ldots, \gamma(t_n), \gamma(1))\}$$

These correspond to the adjoint mappings

$$\Delta^\bullet \times PX \to P^\bullet X \text{ and } \Delta^\bullet \times P_{x,y}X \to P^\bullet_{x,y}X,$$

which are the continuous mappings of cosimplicial spaces used when defining iterated integrals.

18. Iterated Integrals

If one applies the de Rham complex functor to $P^\bullet X$, where X is a smooth manifold, then one obtains a double complex which is quasi-isomorphic to the bar construction on the de Rham complex $E^\bullet(X)$ of X. (If one replaces X^n by $E^\bullet(X)^{\otimes n}$ instead of $E^\bullet(X^n)$, one obtains the bar construction). Elements of this double complex can be evaluated on smooth singular simplices

$$\sigma : \Delta^k \to \|P^\bullet X\|.$$

If σ factors through $PX \hookrightarrow \|P^\bullet X\|$, then the value of $w_1 \otimes \cdots \otimes w_r$ on σ is

$$\langle \int w_1 \dots w_r, \sigma \rangle.$$

19. Chains

One needs a dual theory of chains to get periods. The story with chains is more subtle and goes back to the original work of Adams in the 1950s. Basically, what Adams did was to construct, for each singular simplex $\sigma : \Delta^n \to X$, a map

$$\widehat{\sigma} : I^{n-1} \to P_{x,y}X.$$

One does this by constructing continuous maps

$$\theta_n : I^{n-1} \to P_{0,n}\Delta^n$$

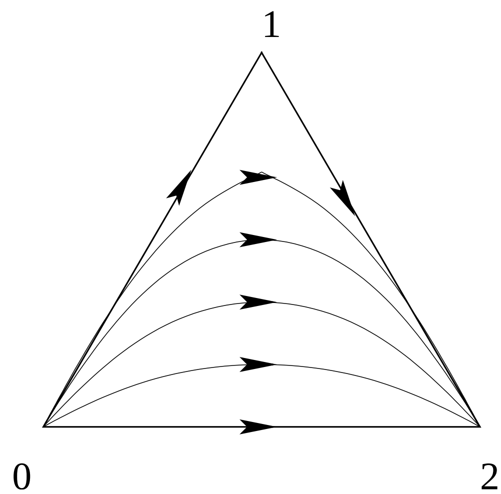
with the property that when $0 < j < n$,

$$\theta_n \circ e_j^0 = P(d^j) \circ \theta_{n-1} : I^{n-2}$$

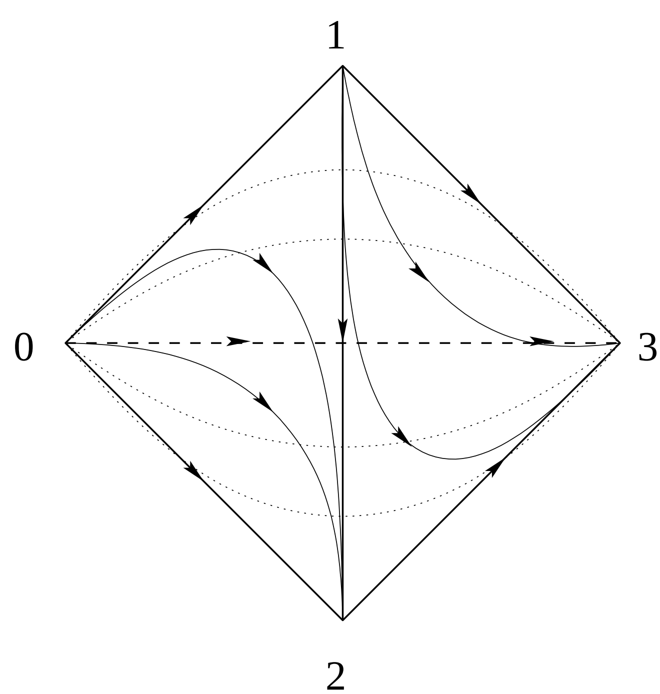
$$\theta_n \circ e_j^1 = \Big(P(f_j) \circ \theta_j\Big) * \Big(P(r_{n-j}) \circ \theta_{n-j}\Big).$$

These can be used to generate a "subalgebra" of the cubical chains on $P_{x,y}X$ whose homology is dual to that $Ch^\bullet(P_{x,y}X)$.

θ_2:



θ_3:



20. Cobar and Motives

The problem, then, is to understand the motivic significance of the cosimplicial cobar construction. More precisely, one would like to show that the cohomology of truncated versions of the bar construction associated to, say, all smooth varieties over a field K, are motives over K in the sense of Levine or Voevodsky.

Partial progress on this problem has been made with Marc Levine for generic points of some varieties, such as $\mathbb{P}^n_{/K}$ when K is a number field. This will be good enough to give a motivic proof of the result obtained with Makoto Matsumoto on the action of the absolute Galois group on the unipotent completion of the fundamental group of $\mathbb{P}^1_{/\overline{\mathbb{Q}}} - \{0, 1, \infty\}$.