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On the Hodge conjecture on some moduli spaces of bundles

Conventions: • work over a field k which can be embedded in $\mathbb{C}$ (e.g. $k=\mathbb{C}$)
• Curve, surface will always mean a smooth projective curve/surface over k.

We would like to look at some Hodge type conjectures:

Hodge conjecture (HC): assume $k=\mathbb{C}$
If X - smooth projective variety $/\mathbb{C}$
$\Rightarrow$ $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ is spanned by codimension p algebraic cycles.

Tate conjecture (TC): assume k = finitely generated extension of $\mathbb{Q}$. Then for any prime ℓ $\Rightarrow$

$$H^{2p}(X_{et} \otimes \bar{k}, \mathbb{Q}_\ell)^{Gal(\bar{k}, k)}$$

is spanned by codimension p cycles defined over k.

Absoluteness conjecture (AC): Roughly this says that if again $k = \mathbb{C}$ then

$$H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

are classes invariant under field automorphisms of $\mathbb{C}$ (note: there are many such automorphisms that one can manufacture via the axiom of choice).

We would like to understand the above conjectures for certain moduli spaces.

Moduli spaces:

Let X be a curve of genus $g \geq 2$. Fix a pair of coprime integers n, d and let

$U_X(n, d)$ = moduli space of stable vector bundles of rank n and degree d over X.

$U_X(n, d)$ is known to be smooth and projective and is also a fine moduli space.

If X is a smooth projective variety and $n > 0$ - fixed, and a vector bundle E over X we can consider

$$Q_n(E) = \{ (F \subset E \text{ -subsheaves of } \mathcal{E}\text{ ; } \mathrm{colength}(F) = n$$

<u>Example</u>: If $E = \mathcal{O}_X \Rightarrow Q_n(E) = \mathrm{Hilb}_n(X)$

<u>Remark</u>:
- $Q_n(E)$ is smooth if X - curve
- $Q_n(\mathcal{O}_X)$ is smooth if X - surface

<u>Question</u>: Is $Q_n(E)$ smooth if X - surface?

<u>Theorem A</u>: If the Hodge (Tate) conjecture hold for all powers of $J(X)$, then HC (TC) holds for $U_X(n,d)$ and AC holds (unconditionally) for $U_X(n,d)$.

<u>Theorem B</u>: If HC (respectively TC) holds for all powers X^N of a surface X, then HC (respectively TC) holds for $\mathrm{Hilb}_n(X)$.

If the Kodaira dimension of $X \leq 0 \Rightarrow$ AC holds for $\mathrm{Hilb}_n(X)$

Corollaries to theorem A:

(a) (Biswas - Narasimhan) If X is a curve with general moduli then HC holds for $\mathcal{U}_X(n,d)$.

(b) HC holds for $\mathcal{U}_X(n,d)$ holds for $g=2$

(c) HC holds for $\mathcal{U}_X(n,d)$ if $J(X)$ is simple and $g=$ prime

(The parts (b) + (c) follow from work of Ribet + Kumar-Murty)

(d) HC holds for $\mathcal{U}_X(n,d)$ if X is the Fermat curve $x^p+y^p+z^p=0$ for p-prime.

Strategy: Cut the spaces into simpler pieces.

Since in the process we will loose projectivity we will need Jannsen's formulation of the HC for quasi-projective varieties:

Let $\mathcal{U}$ be a quasi-projective variety. Use the Borel-Moore homology

$$H_i(\mathcal{U}, \mathbb{Q}) = H^i_c(\mathcal{U}, \mathbb{Q})^\vee$$

This caries a natural (functorial for proper maps) MHS – i.e. the dual of Deligne's MHS on $H^i_C(U, \mathbb{Q})$.

In this setting the HC reads:

$\mathrm{Hom}_{MHS}(\mathbb{Q}(i), H_{2i}(U, \mathbb{Q}))$ is spanned by fundamental classes of i-dimensional algebraic cycles.

<u>Proposition (Jannsen)</u> If $\exists$ a non-singular compactification of U for which the Hodge conjecture holds, then the HC holds for U.

The proofs of theorem A + B are based on the following basic lemmas:

<u>Lemma 1</u>: Assume that X - smooth projective and X can be partitioned into a disjoint union of locally closed varieties each of which satisfies the HC, then so does X.

<u>Lemma 2</u>: If $\pi: X \to Y$ is proper and surjective and Y is a rational homology manifold (e.g. has finite quotient singularities) then HC for X implies HC for Y

<u>Proposition</u> If $T = (\mathbb{C}^\times)^N$ acts on a smooth projective variety X and the HC holds for X^T, then the HC holds for X.

<u>Corollary</u>: If $\dim X^T \leq 3 \Rightarrow$ the HC holds for X.

<u>Remark</u>: • Reduce to a $\mathbb{C}^\times$-action (hence a generic embedding)
• Use Byalynicki-Birula's theorem and apply the previous lemmas to the corresponding decomposition

<u>Two proofs of theorem A</u>:

<u>1st proof</u>: Use the Atiyah-Bott generators

for the cohomology of $U_X(n,d)$.

Write $M := U_X(n,d)$
I = multi-index indexing the Chern monomials on the universal bundle $E \to M \times X$

Then we get a map

$$\begin{array}{ccc} \bigoplus H^\bullet(X^{|I|}) & \xrightarrow{c_I(E)} & H^\bullet(M^{|I|}) \\ & \searrow\!\!\!\searrow & \downarrow \\ & & H^\bullet(M) \end{array}$$

which by Atiyah-Bott is surjective. Then one proves a lemma that the HC for $J(X)^N$ implies HC for X^N + Lemma 2.

<u>2nd proof</u>: let $E = \mathcal{O}(D)^{\oplus n}$
Then $Q_d(E)$ is endowed with a natural $T = (\mathbb{C}^\times)^n$ action.

By Bifet - Ghione - Letizia $\Rightarrow$ the T-fixed pts are symmetric powers of X which are dominated by powers of X which satisfy the HC by hypothesis.

For appropriately chosen N we get

$$Q_N(E)^s = \{ F \subset E \mid F \text{ - stable as bundle} \}$$

so that

$$\begin{array}{c} Q_N(E)^s \\ \downarrow \\ M \end{array}$$

is surjective

and then one compactifies $Q_N(E)^s$ so that

$$\overline{Q_N(E)^s} \to M$$

becomes proper.