Misha Kapranov 04-01-2002

Riemann-Roch theorem for (higher) determinantal gerbes and differentiable manifolds

work in progress with E. Vasserot.
Want to "categorify the Riemann-Roch theorem".

<u>Recall</u>: given a smooth proper morphism $p: X \to S$ of relative dimension n between $\mathbb{C}$-algebraic varieties and $\mathcal{E}$ - holomorphic bundle on X, then by the usual GRR we have

$$c_1(Rp_* \mathcal{E}) := c_1(\det(Rp_* \mathcal{E}))$$

$$= \int_{X/S} \left(ch(\mathcal{E})\, td(T_{X/S}) \right)_{2n+2}.$$

Suppose now that $X \xrightarrow{p} S$ is equipped with a real skeleton $\Sigma \subset X$ along the fibers of p, i.e. we have

$$\begin{tikzcd} X \arrow[r, "p"] & S \\ \Sigma \arrow[u, phantom, "\cup"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

so that $\forall s \in S \Rightarrow \Sigma_s = \pi^{-1}(s) \subset X_s$ is a totally real closed C^∞-submanifold (e.g. looks like $\mathbb{RP}^n \subset \mathbb{CP}^n$).

<u>Remarks</u>: • a priori we put no restrictions on the topology of $\Sigma \subset X$: we only put conditions on the fibrewise embeddings $\Sigma_s \subset X_s$.

• we would like to be able to deduce consequences of RR for $(\Sigma, \mathcal{E}|_\Sigma)$.

• Sato explained that the C^∞-functions on Σ_s can be interpreted cohomologically from the point of view of the embedding $\Sigma_s \subset X_s$.

More precisely Sato shows

$$C^\infty(\Sigma_s) \subset \underset{\substack{\uparrow \\ \text{hyperfunctions.}}}{Hypf(\Sigma_s)} = \underline{H}^n_{\Sigma_s}(X_s, \mathcal{O})$$

Thus we should look for an n-categorical statement.

<u>New setting</u>: $\Sigma \xrightarrow{\pi} S$ C^∞-fibration of smooth compact manifolds of $\mathbb{R}$ dimension $= n$.

$E \to \Sigma$. C^∞ complex vector bundle on Σ.

$\pi_*(E)$ - C^∞ direct image (necessarily of infinite rank).

Now we would like to make sense of $\det(\pi_*(E))$ as some kind of a 'n-gerbe' bound by $\mathbb{C}^\times_S$

<u>Vague proposal</u>: If Σ is a compact C^∞-manifold of $\dim_{\mathbb{R}} = n$ and any C^∞ vector bundle $E \to \Sigma$ $\Rightarrow$ $\Gamma_{C^\infty}(\Sigma, E)$ should give rise to a "determinantal n-gerbe" bound by $\mathbb{C}^\times$.

<u>Examples</u>: (1) $\Sigma = S^1$, $\Gamma_{L_2}(\Sigma, E) =: V$ has a 'polarization':

$\exists$ Sato Grassmannian $Gr(V)$ (Grassmannian of $\frac{\infty}{2}$ dimensional subspaces of V)

$Gr(V)$ parameterizes $L \subset V$ obtained by:

- choosing $\Sigma \subset D \leftarrow$ unit complex disk
- extending E to a holomorphic bundle $\mathcal{E} \to D$
- taking $L = \Gamma_{hol}(D, \mathcal{E})$.

Note: given $L_1, L_2 \in Gr(V) \Rightarrow$
$\exists$ a relative determinant space

$$\det(L_1, L_2) = \det\left(\frac{L_2}{L_1 \cap L_2}\right) \otimes \det\left(\frac{L_1}{L_1 \cap L_2}\right)^{-1}$$

$\Rightarrow$ get a $\mathbb{C}^\times$-gerbe $Det(V)$:

$$\text{Objects} = Gr(V)$$
$$\text{Mor} = \text{Hom}(L_1, L_2) = \det(L_1, L_2) - \{0\}$$

There is a <u>formal version</u>:

Let E be a finite dimensional vector space $\mathbb{C}((t))$ which is a Tate vector space (i.e. $\exists$ a class of commensurable subspaces)

Then $\forall\ L_1, L_2 \subset E$ - commensurable we can define $\det(L_1, L_2)$ purely algebraically by the above formula again.

We also have a formal version for $n = 2$:

Let E be a vector space over $\mathbb{C}((t_1))((t_2))$ i.e. 2-Tate vector space

$\Rightarrow$ for L_1, L_2 we will have that $L_i / L_1 \cap L_2$ is a Tate vector space and so $\mathrm{Det}(L_1, L_2)$ will be a gerbe $\Rightarrow$ $\mathcal{D}et$ a 2-category.

In general if $\mathrm{Det}(\pi_* E)$ has been constructed we will get a class in

$$H^{n+1}(S, \mathcal{O}^{\times}_{C^\infty}) \xrightarrow{d} H^{n+2}(S, \mathbb{Z})$$
$$\rotatebox{90}{$\in$}$$
$$c_1(\mathrm{Det}(\pi_* E))$$

<u>Riemann-Roch problem:</u> describe the above class as

$$\int_{\Sigma / S} (\text{something in } H^{2n+2}(S, \mathbb{Z}))$$

should be the same as in the usual RR.

<u>Example:</u> If M is a $\mathbb{C}$ manifold
Σ_0 - compact C^∞ mfld

$$\mathcal{S} = \mathrm{Maps}(\Sigma_0, M)$$

$$\begin{array}{ccc} \Sigma = \Sigma_0 \times \mathcal{S} & \xrightarrow{ev} & M \\ \pi \downarrow & & \\ \mathcal{S} & & \end{array}$$

$T_{\mathcal{S}} = \pi_*(ev^* T_M)$ - get an n-gerbe on $\mathcal{S}$ with a class in $H^{n+2}(\mathcal{S}, \mathbb{Z})$.

For S^1-fibrations we can construct $c_1(\pi_* E) \in H^3(\mathcal{S}, \mathbb{Z})$.

<u>Thm</u> In $H^3(\mathcal{S}, \mathbb{Q})$ we have

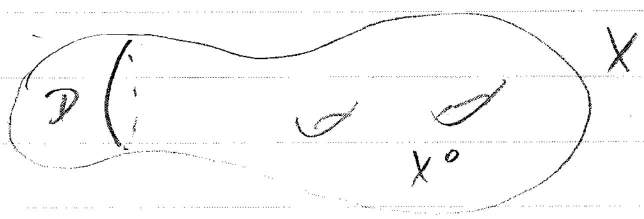
$$c_1(\pi_* E) = \int_{\Sigma/\mathcal{S}} \Big(\underbrace{\tfrac{1}{2} c_1^2(E) - c_2(E)}_{} \Big)$$

truncation of
$(ch(E)\, td(T_{\Sigma/\mathcal{S}}))_4$

<u>Note</u>: $c_1(T_{\Sigma/\mathcal{S}}) = 0$ in $H^2 \otimes \mathbb{Q}$.

<u>Proof for circle fibrations</u>: Use Deligne's version of RR for identifying $\det(R p_* \mathcal{E})$ for a family of smooth complex curves $X \xrightarrow{p} S$ and a holomorphic bundle $\mathcal{E} \to X$.

Idea: include locally
$$\begin{array}{ccc} \Sigma & \subset & X \\ \cap & & \\ D & & \end{array}$$



and extend E to $\mathcal{E}$ on X^0

Then: $\forall$ extension $\mathcal{E}_D$ of E to D we get

- an element of the Sato Grassmannian Gr.
- a holomorphic bundle $\mathcal{E}$ on X

and we only need to identify

$\mathrm{Hom}_{\text{Det gerbe}}$

with $\det(R p_*(\text{of those } \mathcal{E}))$

identified by Deligne

For general $\Sigma \xrightarrow{\pi} S$, $E \to \Sigma$
we cannot identify quite the gerbe
but we can try to identify the
corresponding characteristic class.

We will use a Lie algebra analogue
of the above where formally we
have replaced det by tr.

<u>Note</u>: $\pi_* E$ has a $\mathfrak{g}$-structure
where
$\mathfrak{g}$ = Atiyah algebra of
the fiber
$$\begin{array}{ccc} \Sigma_0 & \subset & \Sigma \\ \downarrow & & \downarrow \\ 0 & \in & S \end{array}$$

i.e.
$$0 \to \mathrm{End}(E_{|\Sigma_0}) \to \mathfrak{g} \to \mathrm{Vect}(\Sigma_0) \to 0$$

Now we can do Chern-Weil theory
via Lie algebra cohomology.

<u>Example</u>: If $n=1$, $\gamma: \mathfrak{g} \to \mathbb{C}$ -trace
$\Rightarrow$ $\forall$ ∇ -connection on $\mathfrak{g}$ $\Rightarrow$
we get a 2-form $\gamma(F_\nabla)$.

More generally if E is a $\mathfrak{g}$-bundle on S and $\gamma \in H^m_{Lie}(\mathfrak{g}, \mathbb{C})$ we get characteristic classes $c_\gamma(E) \in H^{m+1}_{DR}(S, \mathbb{C})$

Indeed: if P is a principal bundle
$$\nabla \rightsquigarrow \omega \in \Omega^1(P) \otimes \mathfrak{g}$$
$$\gamma(\omega) \in \Omega^m(P)$$
and $d(\gamma(m))$ is a lift of a closed $(m+1)$ form on S.

<u>Example</u>: $\mathfrak{g} = \mathfrak{gl}_n$, $\underset{\text{generator}}{\gamma_i \in H^{2i-1}(\mathfrak{g}, \mathbb{C})}$

$\Rightarrow \{\gamma_i\}$ give rise to the usual Chern classes.

In order to construct $\tau \in H^{n+1}_{Lie}\left(\begin{matrix}\text{Atiyah} \\ \text{algebra of} \\ (\Sigma_{S_0}, E)\end{matrix}\right)$

we view
$$\left(\begin{matrix}\text{Atiyah algebra} \\ \text{of } (\Sigma_{S_0}, E)\end{matrix}\right) \subset \mathcal{D}iff(E, E)$$

differential operators.

let F be a vector bundle s.t.
$E \oplus F = \mathbb{C}^N$

$\Rightarrow$ can get our class on $\mathbb{E}$ from some natural class

$$\tilde{\tau} \in H^{n+1}_{Lie}\left(\mathrm{Mat}_\infty \otimes \mathrm{Diff}\left(C^\infty(\Sigma), C^\infty(\Sigma)\right)\right)$$

<u>Recall</u>: $\forall$ algebra R $\Rightarrow$ $H^{Lie}_\bullet(\mathrm{Mat}_\infty(R))$ is freely generated by $HC_\bullet(R)$

But $HC_\bullet\left(\mathrm{Diff}\left(C^\infty(\Sigma), C^\infty(\Sigma)\right)\right) = HC_\bullet(C^\infty(\Sigma))$

Brylinsky, Getzler, Wodzicki

$\Rightarrow$ the embedding

$$\mathrm{Mat}_\infty \otimes C^\infty(\Sigma) \subset \mathrm{Mat}_\infty \otimes \mathrm{Diff}\left(C^\infty(\Sigma)\right)$$

is an isomorphism on $H^\bullet_{Lie}$

On the other hand, there is a well known cocycle of $\mathrm{Mat}_\infty \otimes C^\infty(\Sigma)$ given by

$$\tau(g_0 \cdots g_n) = \int_\Sigma tr(g_0\, dg_1 \cdots dg_n)$$

(Beilinson in 'Residues and adeles' relates this $\tilde{\tau}$ to a trace-type cocycle)

In particular we get $c_1(\pi_* E) \in H^{n+2}(S, \mathbb{C})$

<u>Thm</u>: $$c_1(\pi_* E) = \int_{\Sigma/S} \left(ch(E)\, td(T_{\Sigma/S}) \right)_{2n+2}$$

(The proof uses Feigin - Tsygan "RR for Gelfand - Fuks cohomology").