A. Joyal 04-02-2002

The theory of quasi-categories

quasi-categories : a tool for doing higher category theory

Plan: (1) Definition, motivation
(2) the model structure
(3) left and right fibrations.

Notation:

- Δ : the simplicial category
$= \{ f : [n] \to [m] \mid \text{order preserving maps} \}$

$d_i : [n-1] \to [n]$ omits i

$s_i : [n] \to [n-1]$ repeats i

- a simplicial set is a functor
$X : \Delta^o \to \text{Sets}$. We write $X_n := X([n])$
- $\Delta[n] = \Delta(-, [n])$ - the n-simplex
$\mathcal{S} = [\Delta^o, \text{Sets}]$ the category of simplicial sets
- $\dfrac{\Delta[n] \xrightarrow{x} X}{x \in X_n}$, $\dfrac{[n] \xrightarrow{f} [m]}{\Delta[n] \xrightarrow{f} \Delta[m]}$,

$$\begin{tikzcd} \Delta[m] \arrow[r, "x"] & X \\ \Delta[n] \arrow[u, "f"] \arrow[ur, "xf = X(f)(x)"'] & \end{tikzcd}$$

Yoneda

- $X \in \mathcal{S}$, $a, b \in X_0$ put

$$X_1(a,b) \overset{\text{def}}{=} \{ f \in X_1 \mid a = f d_1,\ b = f d_0 \}$$

we write $f : a \to b$, $1_a = a s_0 \in X_1(a,a)$

$$\begin{tikzcd} {[1]} \arrow[r, "s_0"] \arrow[dr, "1_a"'] & {[0]} \arrow[d, "a"] \\ & X \end{tikzcd}$$

- Cat – the category of small categories

- $N : Cat \to \mathcal{S}$ – the nerve functor

$$\begin{aligned} (NC)_n &= Cat([n], C) \\ &= \{ c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n \mid f \in C \} \end{aligned}$$

<u>Note</u>: N is fully faithful

N has a left adjoint

$$\tau_1 : \mathcal{S} \to Cat$$

$\tau_1 X$ is the <u>fundamental category</u> of X

$\pi_1(X)$ is obtained by freely inverting the arrows in $\tau_1 X$

We also have

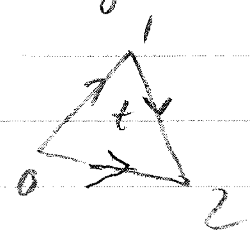
$\tau_0 X$: the set of isomorphism classes in $\tau_1 X$.

<u>Remark</u>: Explicitly $\tau_1 X = FX/\sim$
where

FX : the category of paths in X

$\sim$: congruence of paths :

If



$$t \in X_2 \Rightarrow (td_0)(td_2) \sim (td_1)$$

<u>Example</u>: $\tau_1 \Delta[n] = [n]$.

<u>Note</u>: $\tau_0 X$ and $\pi_0 X$ are <u>different</u>.

Consider now

$$\partial_i \Delta[n] = \mathrm{Im}\left(\Delta[n-1] \xrightarrow{d_i} \Delta[n] \right)$$

$$\partial \Delta[n] = \bigcup_{i=0}^{n} \partial_i \Delta[n]$$

A <u>simplicial sphere</u> in X is a map

$$x : \partial \Delta[n] \to X$$

We will often write $x = (x_0, x_1, \ldots, x_n)$ where $x_i = x d_i$

We will say that a simplicial sphere x <u>commutes</u> if x can be extended to a map $x' : \Delta[n] \to X$.

Consider also $\Lambda^k[n] = \bigcup_{i \neq k} \partial_i \Delta[n]$

A <u>horn</u> in X is a map

$$x : \Lambda^k[n] \to X$$

(we will write $x = (x_0, x_1, \ldots, x_{k-1}, *, x_k, \ldots, x_n)$)

<u>Def:</u> A simplicial set X is called a <u>Kan complex</u> if every horn

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "x'"'] & \end{tikzcd}$$

has a filler x'.

<u>Def</u>: A simplicial set X is a <u>quasi-category</u> if every inner horn

$$\Lambda^k[n] \xrightarrow{x} X \qquad (0<k<n)$$

has a filler.

<u>Examples</u>:

(a) every category is a quasi-category

(b) every Kan complex is also a quasi-category.

<u>Note</u>: A category is a simplicial set X for which every inner horn has a <u>unique</u> filler.

<u>Remark</u>: • Quasi-categories are a generalization both of classical category theory and classical homotopy theory.

• There are new concepts that make sense intrinsically for quasi-categories but do not make sense for Kan complexes: a vertex $a \in X$ in a quasi-category is initial if $\forall\ x : \partial\Delta[n] \to X$ with $x(0) = a \Rightarrow x$ has a filler.

Recall that a map of simplicial sets $X \xrightarrow{f} Y$ is a <u>trivial fibration</u> if for every commutative square

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "d"] & Y \end{tikzcd}$$

we have a diagonal filler d.

<u>Proposition</u>: A simplicial set X is a quasi category $\Leftrightarrow$ the natural map

$$X^{\Delta[2]} \longrightarrow X^{\Lambda^1[2]}$$

is a trivial fibration

<u>Comments</u>: • the meaning of this proposition is that we can choose a composition of arrows in a quasi-category which is homotopically unique. Indeed any section

$$\begin{tikzcd} X^{\Delta[2]} \arrow[r] & X^{\Lambda^1[2]} \arrow[l, bend left, "s"] \end{tikzcd}$$

gives a way of composing two composable arrows in X. Any two such sections are homotopic $\Rightarrow$ every two compositions are homotopic.

An arrow $f: a \to b$ in a quasi-category X is quasi-invertible if it is invertible in $\tau_1 X$

Note that $\forall\, a, b$ we can define also a simplicial hom set $X(a,b)$ in X as the fiber of the map

$$X^{\Delta[1]} \longrightarrow X \times X \qquad \{0,1\} \subset \Delta[1]$$
$$\rotatebox{90}{$\in$}$$
$$(a,b)$$

<u>Proposition</u> If X is a quasi-category, then:

- $X(a,b)$ is Kan.
- $\pi_0(X(a,b)) = \tau_1 X(a,b)$

  ↑ the hom set in $\tau_1 X$

Going back to the notion of a quasi-isomorphism in a quasi-category:

$f : a \to b$ is a quasi-iso iff
$\exists\, g :$ $a \underset{g}{\overset{f}{\rightleftarrows}} b$

and triangles

$$\begin{tikzcd} a \arrow[d, equal, "1_a"'] \arrow[dr, "f"] & \\ a & b \arrow[l, "g"] \end{tikzcd}$$

$$\begin{tikzcd} & b \arrow[dl, "g"'] \arrow[d, equal, "1_b"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

With this understood we have:

<u>Theorem</u>: A quasicategory X is a Kan complex $\Leftrightarrow$ $\pi_1 X$ is a groupoid

<u>Theorem</u> A quasi-category X always contains a largest K complex $Gr(X) \subset X$

<u>Note</u>: • the previous theorem is the analogue of the fact that every category contains a maximal groupoid.

• $Gr(X) \subset X$ explicitly consists of all $x : \Delta[n] \to X$ s.t. $x(i,j)$ is a quasi-equivalence $\forall\ 0 \le i,j \le n$

Let now $Q \subset \mathcal{S}$ be the full subcategory consisting of quasi categories.

<u>Theorem</u>: $\forall X \in Q, \ \forall A \in \mathcal{S} \Rightarrow X^A \in Q$.

In particular Q is cartesian closed. And we can say what an equivalence of quasi-categories is:

A map $f : X \to Y$ in Q is an <u>equivalence</u> if there is a map $g : Y \to X$ and a quasi-isos

$$1_X \overset{\sim}{\to} gf \quad \text{in } X^X$$
$$1_Y \simeq fg \quad \text{in } Y^Y$$

respectively.

<u>Theorem</u>: A map $f : X \to Y$ in Q is an equivalence iff

(a) the map $X(a,b) \to X(fa, fb)$ is a homotopy equivalence $\forall a, b \in X_0$

(b) the map $\tau_0(f) : \tau_0 X \to \tau_0 Y$ is surjective.

<u>Note</u>: (a) should be thought of as f being "homotopy full and faithful"

(b) should be thought of as f being

essentially surjective.

<u>Def:</u> A map of simplicial sets $u : A \to B$ is a <u>weak categorical equivalence</u> if the map

$$X^B \to X^A$$

is an equivalence of quasi categories for all $X \in Q$.

<u>Theorem:</u> The category $\mathcal{S}$ admits a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ with

- $\mathcal{C}$ – the class of all monomorphisms
- $\mathcal{W}$ – the class of categorical weak equivalences
- $\mathcal{F}$ – the class of quasi-fibrations.

<u>Note:</u> • $\mathcal{F} \overset{\text{def}}{=} (\mathcal{C} \cap \mathcal{W})^{\perp}$ ↖ right orthogonal.

• there is no easy explicit description of the class of quasi-fibrations for arbitrary simplicial sets. However quasi-fibrations in Q have a simple concrete description.

<u>Def:</u> A ~~map~~ $f: X \to Y$ is a <u>mid fibration</u> if it is right orthogonal to the horn inclusions

$$\Lambda^k[n] \subset \Delta[n]$$

with $0 < k < n$.

A quasi-fibration in Q is a mid fibration + an extra condition.

More importantly we have notions of

<u>left fibration</u>: same as a mid fibration but we want horns with $0 \le k < n$

<u>right fibration</u>: same as a mid fibration but we want horns with $0 < k \le n$.

<u>Remark</u>: left fibrations (respectively right fibrations) are always quasi-fibrations

Given any $B \in \mathcal{S}$ we can consider a subcategory

$$\mathcal{L}(B) \subset \mathcal{S}/B$$

defined by

$\mathrm{Ob}\,\mathcal{L}(B) = \{ X \xrightarrow{p} B \mid p - \text{left fibration} \}$

$+ \quad \mathcal{L}(B) \subset \mathcal{S}/B$ is full.

Similarly we have $\mathcal{R}(B) \subset \mathcal{S}/B$.
We can put a model structure on $\mathcal{L}(B)$, $\mathcal{R}(B)$ by looking at fiberwise homotopy equivalences of left (or right) fibrations

Weak equivalences in $\mathcal{S}/B$ w.r.t. $\mathcal{L}(B)$ are called ~~[illegible]~~ coinitial equivalences

We have a model structure $(\mathcal{C}, {}_B W, {}_B \mathcal{F})$ on $\mathcal{S}/B$ with ${}_B\mathcal{F}$ being the class of left fibrations.

<u>Theorem</u>: If C is an ordinary category and $[C, \mathcal{S}]$ with its levelwise model structure $\Rightarrow$ there are Quillen equivalences

$$[C, \mathcal{S}] \leftrightarrows \mathcal{S}/C.$$

Thus $\mathcal{S}/B + (\mathcal{C}, {}_B W, {}_B\mathcal{F})$ is a generalization of the category of simplicial presheaves on a category and of their homotopy theory.