INTRODUCTION

This isn't a talk about geometry, at least not explicitly so; nevertheless I think there's a lot of geometry hiding here implicitly; I hope you'll be able to see it.

There will also be more pictures in this talk than any of the talks about geometry we've had so far...

In higher category theory, get pictures like


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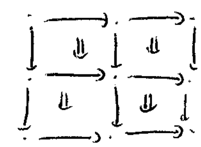
(if you haven't seen this kind of thing before, think of $\cdot$'s as points in some space, $\cdot\to\cdot$'s as paths, $\Downarrow$'s as htpys; you won't go far wrong)


,

(more complicated version of previous one)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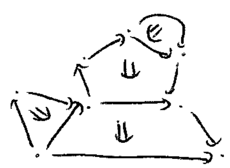
(cubical version)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(different cell shapes again – each 2-cell has many inputs)


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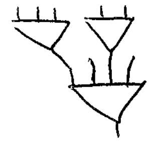

, ...

(same, drawn differently)

and want a language to handle them rigorously.

This language will be an extension of the language of operads and their close cousins, multicategories. I'll start at the very beginning by describing classical operads & multicategories, before I go on to the generalized versions.

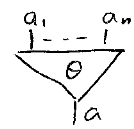
<u>Plan</u>:
1. Classical operads and multicategories
2. Generalized " "
3. Examples.

1. CLASSICAL OPERADS & MULTICATEGORIES

You may be more familiar with operads than multicategories, but I think that for these purposes it's better if I do it the other way round...

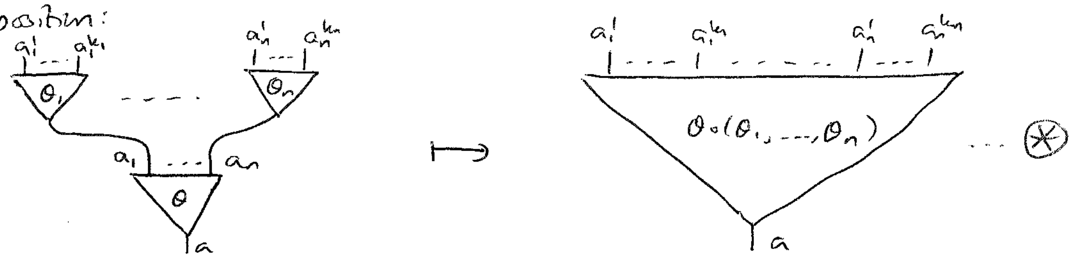
<u>Def-n</u> A <u>multicategory</u> (or "coloured operad") C consists of

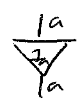
- a class C_0 of "objects"
- for each $n \geq 0$ and $a_1, \ldots, a_n, a \in C_0$, a set $\mathrm{Hom}(a_1, \ldots, a_n; a)$ (an element θ of which is drawn

$$a_1, \ldots, a_n \xrightarrow{\theta} a$$ or 

)

- composition:



- identities: for each $a \in C_0$, an element  $\in \mathrm{Hom}(a; a)$,

satisfying associativity & identity axioms.

Probably the most familiar example is this:

<u>E.g.</u> C_0 is the class of k-vector spaces; $\mathrm{Hom}(A_1, \ldots, A_n; A)$ $= \{\text{multilinear maps } A_1 \times \cdots \times A_n \longrightarrow A\}$. (If you call them coloured operads then the objects are colours, so for each vector space you have to choose a different colour to paint it; since vector spaces form a proper class, you need quite a large palette...)

<u>Def-n</u> An $_\wedge^{\text{(non-symmetric)}}$ <u>operad</u> is a one-object multicategory (i.e. $|C_0| = 1$).

(If you're familiar with operads then you'll see that I'm presenting the most simple kind of operad here — I'll say more about this in a moment.)

Explicitly, an operad O consists of

- a sequence $(O(n))_{n \geq 0}$ of sets ("$O(n) = \mathrm{Hom}(\overbrace{*, \ldots, *}^{n}; *)$")
- a composition map
$$O(n) \times O(k_1) \times \cdots \times O(k_n) \longrightarrow O(k_1 + \cdots + k_n)$$
$$(\theta, \theta_1, \ldots, \theta_n) \longmapsto \theta \circ (\theta_1, \ldots, \theta_n)$$
for each $n, k_i \geq 0$ (see the picture above, but without the labels a_i^j)
- an identity $1 \in O(1)$

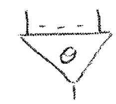
satisfying assoc & id axioms.

<u>E.g.</u> Fix a vector space A: then $O(n) = \mathrm{Hom}(A^{\otimes n}, A)$ has the structure of an operad.

(Can also consider operads & multicategories with symmetries — you can find the axioms by considering the symmetric structure on the monoidal category of vector spaces; indeed, "operad" is usually taken to mean "operad with symmetries" — and in which $O(n)$ or $\mathrm{Hom}(a_1, \ldots, a_n; a)$ has extra structure — i.e. not just a set —, e.g. space, module. Today we ignore these elaborations.)

There are also much less trivial examples of operads & multicategories, e.g. operads play a pivotal role in loop space theory, but I'm just going to discuss the language itself, not the applications.

2. GENERALIZED OPERADS & MULTICATEGORIES

<u>Idea</u>: In an operad / multicategory, an operation  has a <u>1-dimensional sequence</u> of inputs. For higher-dimensional structures, want to consider operations where the input is a <u>higher-dimensional shape</u> (like those drawn at the beginning). E.g. if you want to discuss N-categories, composition will be an operation of this kind. It's not immediately obvious how to formalize this, but the following approach works.

<u>Observation 1</u> A (small) <u>category</u> consists of objects and arrows

$$\begin{tikzcd} & C_1 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\ C_0 & & C_0 \end{tikzcd} \qquad \underbrace{C_1 \times_{C_0} C_1}_{\text{a certain pullback / fibre product}} \xrightarrow{\mathrm{comp}} C_1 \qquad C_0 \xrightarrow{\mathrm{ids}} C_1$$

in <u>Set</u>, satisfying domain (e.g. the domain of a composite of two maps is the domain of the first map), codomain, associativity and identity axioms (all of which can be expressed as commutative diagrams). A <u>monoid</u> is a category C with $|C_0| = 1$; explicitly, a monoid consists of objects and arrows

$$O \ (= \text{“}C_1\text{”}) \qquad O \times O \longrightarrow O \qquad 1 \longrightarrow O$$

satisfying ass & id. (Maps in C = elements of O.)

<u>Observation 2</u> Write $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ for the free-monoid functor : $TX = \coprod_{n \geq 0} X^n$.
Then a <u>multicategory</u> consists of objects and arrows

$$\boxed{\begin{tikzcd} & C_1 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\ TC_0 & & C_0 \end{tikzcd} \qquad C_1 \times_{TC_0} TC_1 \xrightarrow{\mathrm{comp}} C_1 \qquad C_0 \xrightarrow{\mathrm{ids}} C_1} \qquad (\ast)$$

in <u>Set</u>, satisfying (co)domain, ass & id axioms (again expressible diagrammatically). (Think: $C_1 = \coprod_{a_1, \ldots, a_n, a \in C_0} \mathrm{Hom}(a_1, \ldots, a_n; a)$; $C_1 \times_{TC_0} TC_1$ is the set of diagrams on the LHS of $(\ast)$.) An <u>operad</u> is a multicategory C with $|C_0| = 1$.

We're nearly ready to generalize Observations 1 & 2, but first we need one piece of categorical vocabulary.

<u>Def'n</u> A <u>monad</u> (or "triple") on a category $\mathcal{E}$ is a monoid in the monoidal category $(\mathrm{End}(\mathcal{E}), \circ, 1_{\mathcal{E}})$: that is, it's a functor $T : \mathcal{E} \to \mathcal{E}$ with natural transf's $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{E}} \to T$ satisfying ass & id axioms.

Category theorists think of monads as the same thing as algebraic theories: the principal examples of monads come from taking the free algebraic structure of a given kind.

<u>E.g.</u>
- $\mathcal{E} = \mathbf{Set}$, $T =$ free-monoid
  (If you haven't seen monads before then you can ignore η and μ – just think of a monad on $\mathcal{E}$ as a functor from $\mathcal{E}$ to $\mathcal{E}$.)
- $\mathcal{E} = \mathbf{Set}$, $T =$ identity, η & μ identities.

Now the main def'n!

<u>Def'n</u> (Burroni) Let $\mathcal{E}$ be a category and $T = (T, \mu, \eta)$ a monad on $\mathcal{E}$; suppose that "finite limits behave" (I won't make this precise here). A <u>T-multicategory</u> consists of objects & arrows (†) in $\mathcal{E}$, satisfying (co)domain, ass & id axioms. A <u>T-operad</u> is a T-multicategory C in which $C_0 = 1$ (i.e. C_0 is a terminal object).

Rest of the talk will be examples and explanation of this def'n.

3. EXAMPLES

(a) Let $\mathcal{E} = \mathbf{Set}$ & $T = \mathrm{id}$: then

a T-multicaty is exactly a category
a T-operad " " " monoid

(Observation 1).

(b) Let $\mathcal{E} = \mathbf{Set}$ & $T = $ free-monoid: then

a T-multicaty is exactly a multicaty
a T-operad " " " operad

(Observation 2).

(c) Fix $N \geq 0$. An <u>N-globular set</u> is a diagram

$$A_N \underset{t}{\overset{s}{\rightrightarrows}} A_{N-1} \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} A_0$$

of sets and functions such that $s \circ s = s \circ t$ & $t \circ s = t \circ t$. Pictures:

$\bullet a \in A_0$ "0-cell" $\qquad$ $a \bullet \xrightarrow{f} \bullet b \in A_1$ "1-cell" $\qquad$ $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd} \in A_2$ "2-cell"

Any (strict) N-caty has an underlying N-globular set; conversely, (these ones are easy!) can form the free N-category on any N-globular set. So we obtain a monad

$$T = (\text{free } N\text{-category})$$

on the category

$$\mathcal{E} = (N\text{-globular sets}).$$

E.g. writing $1 = (\{*\} \rightrightarrows \cdots \rightrightarrows \{*\})$ (the terminal N-globular set),

$(T1)_0 = \{\bullet\}$

$(T1)_1 = \{\bullet,\ \bullet\!\to\!\bullet,\ \bullet\!\to\!\bullet\!\to\!\bullet,\ \ldots\} \cong \mathbb{N}$ (an element is just k arrows stuck end-to-end)

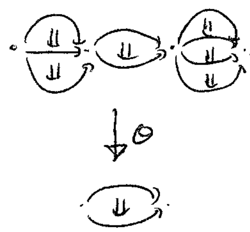
$(T1)_2 = \{$diagrams like  $\}$. (formal pastings again)

So that's $\mathcal{E}$ and T; what are T-multicategories and T-operads?
For simplicity I'll just describe T-operads.

A T-operad ("globular operad") O consists of

$$\begin{tikzcd} & O \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\text{(unique)}"] & \\ T1 & & 1 \end{tikzcd}$$

- for each $p \in (T1)_d$, a set $O(p)$ (e.g. we draw $\theta \in O(q)$ as



and think of θ as an operation which takes input of this shape and produces output of this shape)

- compositions, e.g.

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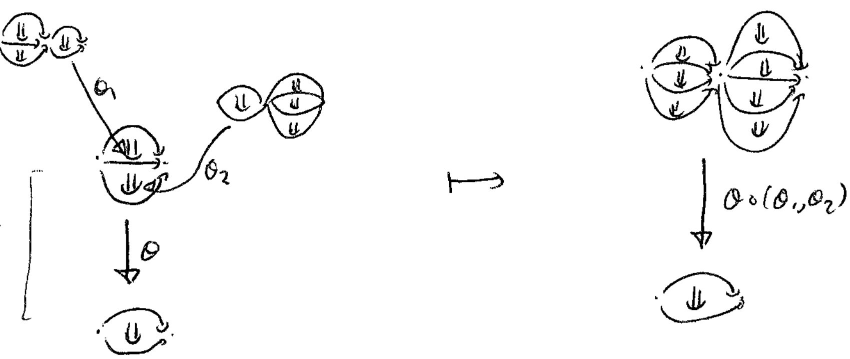
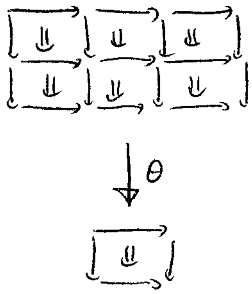


- identities

satisfying assoc & id axioms.

(For any $\mathcal{E}$ and T, can easily & simply define ~~an~~ algebras for a T-multicaty; this extends usual notion of algebra for an operad. Can construct a certain globular operad O & define a weak N-category to be an algebra for O — one amongst many possible defns, but with this language it's very quick to state.)

(d) There's a cubical analogue of (c), where operations look like



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take $\mathcal{E}$ = (N-cubical sets) = (functors $(\bullet \rightrightarrows \bullet)^N \longrightarrow \mathbf{Set}$),

T = (free N-tuple caty).

↑ = "cubical N-caty" — double caty when N = 2

(e) Take $\mathcal{E} = \mathbf{Set}^{\mathbb{N}}$ and T = (free operad). Then
in the sense of §1

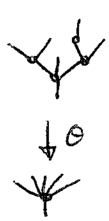
$(T1)_n = \{n\text{-leafed trees}\}$,

as an element of T1 is a formal composite of the shapes

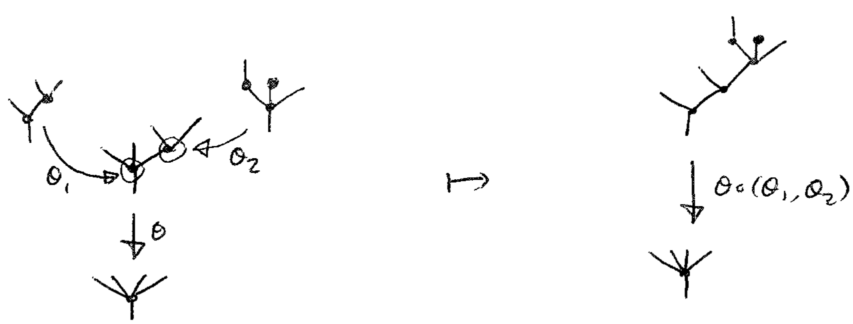
(Again, for simplicity just look at T-operads.) So a T-operad consists of

- for each tree τ, a set $O(\tau)$ (e.g. we draw $\theta \in O($  $)$

as



- compositions, e.g.



- identities.

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(f) Iterate this thought: for each $n \geq 0$, define a category $\mathcal{E}_n$ and a monad T_n on $\mathcal{E}_n$:

- $\mathcal{E}_0 = \mathbf{Set}$, $\quad T_0 = \mathrm{id}$
- $\mathcal{E}_{n+1} = \mathcal{E}_n / T_n 1 = \Big($diagrams $\begin{tikzcd} & ? \arrow[dl, "?"'] \arrow[dr] & \\ T_n 1 & & 1 \end{tikzcd}$ $\Big)$, — this bit's trivial

$T_{n+1} =$ (free T_n-operad). (which makes sense as any T_n-operad has an underlying diagram of that form)

So

$\mathcal{E}_0 = \mathbf{Set}$, $\qquad T_0 = \mathrm{id}$, $\qquad T_0$-operad = monoid

$\mathcal{E}_1 = \mathbf{Set}/T_0 1 \cong \mathbf{Set}$, T_1 = free-monoid, T_1-operad = operad

$\mathcal{E}_2 = \mathbf{Set}/\mathbb{N} \simeq \mathbf{Set}^{\mathbb{N}}$, T_2 = free-operad, T_2-operad: as in (e).

Can see that $\forall n$, there's a set $\mathbb{O}_n$ such that $\mathcal{E}_n \simeq \mathbf{Set}^{\mathbb{O}_n}$.

$\mathbb{O}_n$ is the set of <u>n-opetopes</u> (or n-dimensional opetopes; originally defined by Baez & Dolan in a rather different way).

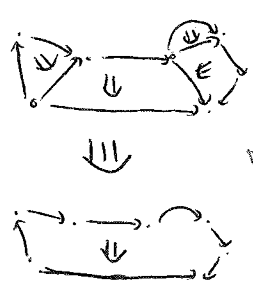
Pictures:

$\mathbb{O}_0 = \{\bullet\}$

$\mathbb{O}_1 = \{\bullet \longrightarrow \bullet\}$

$\mathbb{O}_2 = \{$  $, \ldots \} \cong \mathbb{N}$

(we're sticking together a string of 1-opetopes then turning it into a 2-dimensional shape by adding a single arrow $\longrightarrow$ at the bottom)

$\mathbb{O}_3 = \{$diagrams like  $\} \cong \{$trees$\}$

view as a 3-dimensional figure with 4 curved faces on the top & 1 flat face on the bottom

etc!

Opetopes & "face maps" form a category $\mathbb{O}$; can define a weak N-category as a functor $\mathbb{O}^{op} \longrightarrow Set$ satisfying horn-filling conditions (cf. Joyal's talk). This is essentially the approach of Baez & Dolan's definition.