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## Formal deformations of sheaves of algebras

$X$ – separated scheme $/k$  char $k = 0$

**Problem**: describe the formal deformations of $X$.

Standard approach: fix $(R, m)$ where
- $R$ – artinian local $k$-algebra
- $m \triangleleft R$ – the maximal ideal
- $k = R/m$ – the residue field

Consider $Def^0_X(R)$ = iso classes of $R$-flat schemes $\tilde{X} \to \operatorname{Spec} R$ with closed fiber $X$.

This gives a functor

$$Def^0_X : \operatorname{art}(k) \to \operatorname{Set}$$

and we want to represent this functor.

More precisely we would like to find a complete local ring $R_\infty$ s.t.

$$Def^0_X(R) = \operatorname{Hom}_{\substack{\text{local}\\ k\text{-alg}}}(R_\infty, R)$$

Note: • $R_\infty$ may not exist
• even if $R_\infty$ exists, its formal spectrum need not be formally smooth.

Examples:

(1) Let $X$ be a smooth scheme
$R = k[\varepsilon]/\varepsilon^2$
To calculate $\mathrm{Def}^0_X(R)$ we write

$$X = \bigcup_i U_i \qquad U_i - \text{affine}$$

Then $\tilde{U}_i = U_i \times \operatorname{Spec} R$ necessarily and so $\tilde{X} = \bigcup_i \tilde{U}_i$ is uniquely determined by the gluing data

$$\theta_{ij} : \tilde{U}_i \cap \tilde{U}_j \to \tilde{U}_j \cap \tilde{U}_i .$$

Explicitly
$$\theta_{ij} \in \mathrm{Aut}_R(R \otimes \mathcal{O}_{ij}) = \mathrm{Der}(\mathcal{O}_{ij}, \mathcal{O}_{ij})$$

and $\{\theta_{ij}\} \in H^1(X, T_X)$.

This gives an identification (Kodaira-Spencer)

$$\mathrm{Def}^0_X(R) = H^1(X, T_X)$$

If $R_\infty$ exists $\Rightarrow$ we will have to have an identification

$$Def^0_X(R) = H^1(X, T_X) = Maps(R_\infty, R) = (m_{R_\infty}/m_{R_\infty}^2)^\vee.$$

(2) Let $X$ be an affine (not necessarily smooth) scheme

Let $X = Spec(A)$, $R = k[\varepsilon]/\varepsilon^2$

To compute

$$Def^0_X(R)$$

we begin by taking a free algebra resolution

$$\mathcal{P} \longrightarrow A$$

and forming

$$T_A := Der(\mathcal{P}, \mathcal{P})$$

graded derivations of $\mathcal{P}$.

<u>Note</u>:
- $\mathcal{P}$ is a graded algebra with free terms.
- $Der(\mathcal{P}, \mathcal{P})$ is naturally a Lie algebra equipped with a differential coming from $\mathcal{P}$.

It turns out that $Def^0_A(R) = H^1(T_A)$

In fact $H^1(T_A) \cong Harr^2(A, A)$
(Harrison or Andre-Quillen cohomology of $A$)

Example (Formal Lie theory):

If $X$ = smooth affine over $k$,
then
$$Def^0_X = *$$
but we can still look at

$Def^1_X$ = the groupoid of all possible deformations of $X$ over local artin $k$-algebras.

Then $Def^0_X(R) = \pi_0(Def^1_X(R))$
but
$Def^1_X(R)$ is not trivial

Let $X = Spec(A)$
$R = k[\varepsilon]/\varepsilon^2$

Then $Def^1_X(R) = Aut_R(X \times Spec\, R)$
= the value of the formal group corresponding to $Der(A)$ at $R$

Explicitly

$$\mathrm{Def}^1_X(A) = \exp(m \otimes \mathrm{Der}(A))$$

nilpotent group with a multiplication given by the BKH formula.

<u>Generalization</u>: Given a $\mathfrak{g}$ - dg Lie algebra over $k$ we can define the Deligne - groupoid functor
$$\mathrm{Del}_{\mathfrak{g}} : \mathrm{art}(k) \to \text{groupoids}$$
where
$$\mathrm{Ob}\left(\mathrm{Del}_{\mathfrak{g}}((R, m))\right) =$$
$$= \{ z \in (m \otimes \mathfrak{g})^1 \mid dz + \tfrac{1}{2}[z, z] = 0 \}$$

$$\mathrm{Hom}(z, z') = \{ \gamma \in \exp(m \otimes \mathfrak{g})^0 \mid z' = \gamma(z) \}$$

With this in mind we can go back to the above examples and identify a dg Lie algebra $\mathfrak{g}$ which controls the corresponding deformation problem

<u>In Example 2:</u> We had $X = \operatorname{Spec} A$
$\mathcal{R} \to A$ free resolution

So - can take
$$T_A = \operatorname{Der}(\mathcal{R}, \mathcal{R})$$ dg Lie algebra

and then
$$\operatorname{Def}^1_X = \operatorname{Del}_{T_A}$$

<u>In Example 1:</u> We had $X$ - smooth
we have $T_X$ - tangent bundle
and one can define a dg Lie algebra structure on $R\Gamma(X, T_X)$

let $\mathbb{T}_X \overset{\text{def}}{=} R\Gamma(X, T_X)$ + Lie algebra structure

(the tangent Lie algebra for $X$)

Then again we have an isomorphism

$$\operatorname{Def}^1_X = \operatorname{Del}_{\mathbb{T}_X}.$$

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There are two extensions of the Deligne functor Delg which are useful.

More precisely, given $\mathfrak{g}$ a dg Lie algebra we can construct extensions of $\mathrm{Def}_{\mathfrak{g}}$ to functors

$$\mathrm{art}(k) \longrightarrow \text{Kan simplicial sets}$$

$$\mathrm{art}(k) \longrightarrow \text{simplicial groupoids}$$

These two extensions are useful since they naturally will lead to <u>uniqueness</u> for the representing object $R_\infty$.

<u>Remark</u>: In fact with these extensions we will need to force $R_\infty$ in $\mathrm{dg\,art}(k)$

Next we would like to merge the previous two examples in a way that will solve the problem for arbitrary $X$ - not necessarily smooth or affine.

let $X$ be a site
$\mathcal{O}$ a presheaf of rings on $X$
$C(X^\wedge, \mathcal{O})$ - presheaves of complexes of $\mathcal{O}$-modules

**Theorem** There is a closed model category structure on $C(X^{\wedge}, \mathcal{O})$ where:

(1) the weak equivalences are maps of complexes of presheaves

$$f : M \to N$$

s.t. the corresponding sheafification

$$h^i(f)^a : (H^i M)^a \to (H^i N)^a$$

is an isomorphism on cohomology sheaves

(2) cofibrations are generated by maps of the following form:

given: • $M$ - presheaf

• $z \in M(U)$, for some open $U$ s.t. $dz = 0$

• $M' = M$ + free generator $e$ s.t. $de = z$

we take $M' \to M$ - cofibration.

(3) $f : M \to N$ is a fibration if for hypercovering $\varepsilon : V_\bullet \to U$ in $X$

the diagram

$$\begin{tikzcd} M(U) \arrow[r] \arrow[d, "f(U)"'] & \check{C}(V_\bullet, M) \arrow[d] \\ N(U) \arrow[r] & \check{C}(V_\bullet, N) \end{tikzcd}$$

is homotopy cartesian.

<u>Theorem</u>: The category of $k$-algebras over $X$ admits a closed model structure given by the same (1), (2), (3) provided that:

- $k \supset \mathbb{Q}$
- $X$ has enough points

Equipped with these two theorems we can proceed solving our deformation problem. In fact we can solve a more general deformation problem.

Let $X$ be a site with enough pts
$k$ - field of characteristic $0$
$A$ - sheaf of $k$-algebras on $X$

We can describe the deformations of $A$ over $X$ as follows:

- choose a presheaf of dg algebras $\mathcal{L}$ on $X$ s.t.

$\mathcal{L} \underset{w.e.}{\sim} A$
$\mathcal{L}$ - fibrant and cofibrant

- take the sheaf of dg Lie $k$-algebras $\mathrm{Der}(\mathcal{L}, \mathcal{L})$

– take $T_A = R\Gamma(\mathcal{D}er(\mathcal{P}, \mathcal{P}))$

<u>Theorem</u>: $\mathrm{Def}_A = \mathrm{Def}_{T_A}$ provided that

(1) char $k = 0$, $X$ has enough points

(2) if $A$ is an algebra over an operad $\Rightarrow$ this operad is non-positively graded

(3) $H^i(U, A) = 0$ for $i > 0$ $\forall\, U \in X$

(4) finiteness conditions on $X$

(5) $U \times V \in X$ $\forall\, U, V \in X$

<u>Example</u>: Given a scheme $S/k$
$X$ = site of affine open subschemes in $S$
$A = \mathcal{O}_X$.

Then (3) holds trivially
(4) holds if $X$ - quasi compact
(5) holds if $X$ - quasi separated.

<u>Remark</u>: $\mathrm{Def}_{T_A}(R) := \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{L}(R^\vee), T_A)$
where:

- $R^\vee$ – coalgebra

- $\mathcal{L}(R^\vee)$ the dg-Lie algebra corresponding to $R^\vee$ via the Quillen functor $\mathcal{R}$
- $\underline{\mathrm{Hom}}$ the simplicial set of maps in the simplicially enriched category of dg Lie algebras
- $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{L}(R^\vee), TA)$

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{L}(R^\vee), TA) = \underline{\mathrm{Hom}}\left(\begin{array}{c}\text{cofibrant}\\ \text{replacement}\\ \text{of } \mathcal{L}(R^\vee)\end{array}, \begin{array}{c}\text{fibrant}\\ \text{replacement}\\ \text{of } TA\end{array}\right)$$