G. Faltings 04-03-2002

p-adic Hodge theory I

p-adic Hodge theory investigates the relation

(étale cohomology) $\longleftrightarrow$ (crystalline or de Rham cohomology)

First we need to introduce both sides of the game:

Let K be a "p-adic" field:

- fix a dvr V with a uniformizer π
- assume $k = V/\pi V$ is a perfect field of characteristic $p > 0$
- take $K = V[\frac{1}{\pi}]$ - char 0 and V-complete
- let $W(k) = V_0 \subseteq V$ - finite, (Witt vectors of k) $f(\pi) = 0$, $f(T) = T^e + a_1 T^{e-1} + a_e \in V_0[T]$ p-divides a_i, $a_e = p \cdot$ unit.

If X is a scheme over K we can look at:

- étale cohomology $H^i(X \otimes_K \bar{K}, \mathbb{Z}_p / \mathbb{Q}_p)$ with the natural $Gal(\bar{K}/K)$ action
- de Rham cohomology of X: $\mathbb{H}^i(X, \Omega^{\cdot}_{X/K})$ (X-smooth) has a Hodge filtration

• crystalline cohomology: if X is smooth over V as well, then

$H^i_{DR}(X/V)$ depends only on $X \otimes V/pV$

and we can define it to be

$H^i_{cr}(X \otimes V/pV, V)$

<u>Note</u>: the ideal (pV) has divided powers (recall that an ideal I has divided powers if $x \in I$ implies $\frac{x^n}{n!} \in I$ for all n).

<u>The Gauss-Manin connection</u>: If $X \xrightarrow{f} \mathcal{S}$ is smooth, then $Rf_* X/\mathcal{S}$ has an integrable connection.

Note that if $\mathcal{E}/\mathcal{S}$ with ∇ - integrable, then the fibers of $\mathcal{E}$ over $s_1, s_2 \in \mathcal{S}$ s.t. $s_1 \equiv s_2 \mod p$ are isomorphic by the Taylor series

$$pr_2^*(e) = \sum_I pr_1^*(\nabla^I_{\partial/\partial t}\, e) \cdot \frac{(t_2 - t_1)^I}{I!}$$

So if we have divided powers we can identify the cohomology of X_{s_1} and X_{s_2} provided that $s_1 \equiv s_2 \mod p$.

Usually: after $\otimes_V K$ these Taylor series converge for $\|t_2 - t_1\| < 1$.
Thus (de Rham cohomology) $\otimes\, \mathbb{Q}_p$ depends only on (X mod π).

One can also consider the crystalline cohomology of $(X \otimes_V k / V_0)$ with coefficients in either V_0-modules or K_0-vector spaces

V_0 has a Frobenius σ $\Rightarrow$ get a map

$$\mathrm{Frob} : X \otimes k \to (X \otimes_V k) \otimes_\sigma k$$

which via pulling back induces a Frobenius action on crystalline cohomology:

$$\mathrm{Frob}^* : H^i_{cr}(X \otimes k / V_0) \otimes_\sigma V_0 \to H^i_{cr}(X \otimes k / V_0).$$

The fundamental fact here is the

Hyodo-Kato isomorphism:

$$\underset{\substack{\uparrow \\ \text{has a Frobenius action}}}{H^i_{cr}(X \otimes k / V_0)} \otimes_{V_0} K \;\tilde{\to}\; \underset{\substack{\uparrow \\ \text{has a Hodge filtration.}}}{H^i_{DR}(X_K / K)}$$

<u>heuristic proof:</u> Assume that X lifts to a family

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathrm{Spec}(V_0[[t]]) \\ \cup & & \uparrow \\ X & \longrightarrow & \mathrm{Spec}(V) \end{array}$$

where $V_0[[t]] \twoheadrightarrow V$
$t \to \pi$

Consider $H^i_{DR}(\mathfrak{X}/V_0[[t]])$. Since $V_0[[t]]$ has a Frobenius: $t \mapsto t^p$ $\Rightarrow$ this cohomology has a semilinear Frobenius modulo p. This Frobenius becomes an isomorphism after inverting p $\Rightarrow$ we get a Frobenius crystal $\mathcal{E}$ on $V_0[[t]]$ which is constant over $K_0\{\{t\}\}$ (power series converges on open unit disks).

Now $\mathcal{E}$ is such that $\mathcal{E}_0 = \mathcal{E}/t\mathcal{E}$ is a K_0-vector space and hence we may choose

$$\tilde{F} : \mathcal{E}_0 \otimes_{K_0} K_0\{\{t\}\} \to \mathcal{E}$$

which lifts the identity mod t

Then $\Phi^* \tilde{F} = \Phi^*_0 \circ F \circ \Phi_0^{-1} \equiv \tilde{F} \mod t$

$\Rightarrow$

$$\lim_{n\to\infty} \Phi^{n,*}(\tilde{F}) = \tilde{F} + \sum_{n=0}^{\infty} \underbrace{(\Phi^n)^*(\underbrace{\Phi^*\tilde{F} - \tilde{F}}_{\substack{\parallel\\ 0 \mod t}})}_{}$$

$\equiv 0 \mod t^{p^n}$

denominator $\leq p^{\text{const}\cdot n}$

$\nearrow$

power series converges as $\sum \frac{t^{p^n}}{p^n}$

Thus we get an isomorphism

$$\mathcal{E}_0 \otimes_{K_0} K_0\{\{t\}\} \xrightarrow{\sim} \mathcal{E}$$

and fibre at $t=0$ = crystalline cohomology
fibre at $t=\pi$ = de Rham cohomology over K.

Since $\mathcal{E}$ is a constant crystal we deduce the Hyodo-Kato isomorphism.

<u>Example</u> Abelian varieties (everything generalizes to p-divisible groups).

Let A/V - an abelian variety
Consider the Tate module of A:

$$T_p(A) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A(\bar{K}))$$
$$\cong \mathbb{Z}_p^{2g} \text{ with } \mathrm{Gal}(\bar{K}/K) \text{ action}$$

<u>Note</u>: $H_{1,\acute{e}t}(A) = T_p(A)$.

On the crystalline side consider the universal vector extension

$$0 \to H^1(A, \mathcal{O}_A)^\vee \to EA \to A \to 0$$

Then
$$\mathrm{Lie}(EA) = H_{1,DR}$$
Also
EA depends only on the reduction of A modulo a divided power ideal I.

<u>Remark</u>: A heuristic proof of the last statement: consider the module extension of $A(R/I)$:

$$\begin{array}{c} \ker(EA(R) \longrightarrow A(R/I)) \\ \downarrow \cong \log \\ \ker(\mathrm{Lie}(EA) \longrightarrow \mathrm{Lie}(A) \bmod I) \end{array}$$

In this case the Hyodo-Kato isomorphism can be described in elementary terms: We want to compare

$$H_{1,DR}(A/V) = H_{1,cr}(A \otimes V/pV, V)$$

and

$$H_{1,cr}(A \otimes k / V_0)$$

but

$$(A \otimes k) \otimes_k V/pV \quad \text{and} \quad A \otimes V/pV$$

become isomorphic if we apply a sufficiently high power of the Frobenius.

Galois cohomology of $\bar{K}$

We want to compute

$$H^{\bullet}_{cont}(Gal(\bar{K}/K), \hat{\bar{K}}(i))$$

Tate: Cohomology is trivial for unramified extensions

But we have

$$K \subset \underbrace{K(\mu_{p^\infty}) \subset \bar{K}}_{\text{almost unramified}}$$

in the sense that if $K \subseteq L$ finite extension and if $K_n = K(\mu_{p^n})$, $L = LK_n$

$$\Rightarrow \quad \mathrm{val}(\mathrm{discr}(L_n | K_n)) \xrightarrow[n\to\infty]{} 0$$

In particular this implies that

$$H^i_{cont}(\mathrm{Gal}(\bar{K}/K), \hat{\bar{K}}(j)) = H^i_{cont}(\mathrm{Gal}(K_\infty/K), \hat{K}_\infty(j))$$

$$= \begin{cases} 0 & \text{if } j \neq 0 \\ K & \text{if } j = 0,\ i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Also if $\bar{V}$ = integral closure of V in $\bar{K}$

$\Rightarrow$ Fontaine has computed $\Omega_{\bar{V}/V}$:

$$\Omega_{\bar{V}/V} = \bar{K}/\mathfrak{p}^{-1}\bar{V}(1)$$

$+$ $0 \neq \mathfrak{p} \in \bar{V}$ and has a valuation $\frac{p-2}{p-1} + v_p(\mathrm{discr})$.

Also we have a map

$$\begin{array}{ccc} d\log\zeta : \mu_{p^\infty} & \longrightarrow & \Omega_{\bar{V}/V} \\ \| & & \| \\ \mathbb{Q}_p/\mathbb{Z}_p(1) & & \bar{K}/\mathfrak{p}^{-1}\bar{V}(1) \end{array}$$

We can use all of these to construct comparison maps:

Fontaine: let A/V – abelian variety and:

- let $\rho \in T_p(A)$ corresponds to a sequence of torsion points $x_n \in A(\bar{V})[p^n]$
- let $\alpha \in \Omega_A = \mathrm{Lie}(A)^\vee$
  $x_n^*(\alpha) \in \Omega_{\bar{V}/V}$

we get a pairing

$$T_p(A) \times \Omega_A \to \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega_{\bar{V}/V})$$
$$\| $$
$$\rho^{-1} \cdot \hat{\bar{V}}(1)$$

Tate: If $\rho \in T_p(A)$ over $\bar{V}$ gives a compatible family of maps $A^\vee[p^n] \to \mu_{p^n} \subset \mathbb{G}_m$

Then pulling back $\frac{dt}{t}$ via these maps and glueing together gives an element in $\Omega_{A^\vee} \otimes \hat{\bar{V}}$ i.e. we obtain a map

$$T_p(A) \to \Omega_A \otimes \hat{\bar{V}} = (\text{dual of } H^1(A, \mathcal{O}_A)) \otimes \hat{\bar{V}}$$

Thus putting the Fontaine + Tate together we get

$$\rho^{-1}\mathrm{Lie}(A)\otimes\hat{\bar{V}}(1) \subset T_p(A)\otimes\hat{\bar{V}} \to \Omega_{A^\vee}\otimes\hat{\bar{V}}$$

which induces an isomorphism

$$T_p(A)\otimes\hat{\bar{K}} \cong \left(\mathrm{Lie}(A)\otimes\hat{\bar{K}}(1)\right)\oplus\left(\Omega_{A^\vee}\otimes\hat{\bar{K}}\right)$$

Hodge-Tate decomposition