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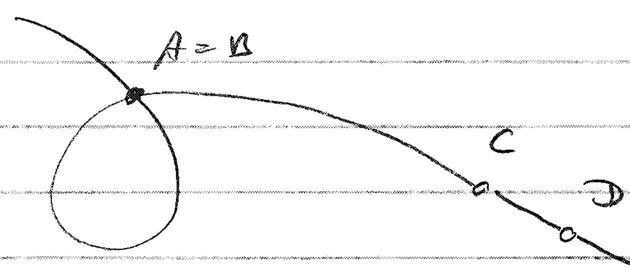
MHS and fiber bundles over $\mathbb{P}^2$

Plan: 0. 1st motivation
1. filtrations and the Rees construction
2. MHS
3. Relative constructions

0. Calculus of MHS

Let X be a curve of genus 0 with two punctures C, D and a node given by identifying $A, B \in \mathbb{P}^1$.

$X = \mathbb{P}^1$



Consider

$$\begin{tikzcd} X' \arrow[r, hook, "\bar{j}"] \arrow[d, "r"'] & \bar{X}' \arrow[d, "\bar{r}"] \\ X \arrow[r, hook, "j"'] & \bar{X} \end{tikzcd}$$

where X' = normalization of X
$\bar{X}$ = compactification of X

$\bar{X}'$ = the normalization of $\bar{X}$.

Deligne in Hodge III showed that

$$H^1(X,\mathbb{C}) = \mathbb{H}^1(\bar{X}, [\mathcal{O}_{\bar{X}} \xrightarrow{d} \Omega_{\bar{X}}(\log S)])$$

(where $S = \bar{X} - X = \{C, D\}$) and that

$$F^1 H^1(X, \mathbb{C}) = \left\langle \frac{du}{u-C}, \frac{du}{u-D} \right\rangle_{\mathbb{C}}$$

We would like to understand the relative position of F^1 and $\bar{F}^1$.

$H_1(X, \mathbb{R}) = \langle \gamma_1, \gamma_0 \rangle_{\mathbb{R}}$ $\qquad \gamma_0 =$

$\gamma_1 = A \to B$

$$\Rightarrow \int_{\gamma_0} F^1 = 2\pi i$$

$$\int_{\gamma_1} F^1 = \log(A, B, C, D)$$

$\mathbb{P}GL(2)$ $\quad A \to 0$ $\quad D \in \mathcal{M}_{0,4}(\mathbb{P}^1) \cong \mathbb{C} - \{0,1\}$

$B \to 1$

$C \to \infty$

Thus the period matrix for F^1 is $\left(1, \log\left(\frac{D}{D-1}\right)\right)$

the period matrix for F^1 is given by
$\left(1, \log\left(\frac{D}{D-1}\right)\right)$

So $D \in \left(\mathrm{Re} = \frac{1}{2}\right)$ is a special line.

Now consider
$$h^{pq} = gr_F^p \, gr^W_{p+q} \, H^1(X, \mathbb{C})$$

$$s^{pq} = gr_F^p \, gr_{\bar{F}}^q \, H^1(X, \mathbb{C})$$

$\Rightarrow$ h^{pq}

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

s^{pq} if $D \in \left(\mathrm{Re} = \frac{1}{2}\right)$

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

s^{pq} if $D \notin \left(\mathrm{Re} = \frac{1}{2}\right)$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

To understand the relationship between $gr_F gr^W$ and $gr_F gr_{\hat F}$ we will associate vector bundles on $\mathbb{P}^2$ to a MHS.

<u>Remark</u>: The construction is inspired by:

• "Mixed twistor structures", C. Simpson associates a $\mathbb{G}_m$ equivariant bundle $\xi(V, F, \hat F)$ on $\mathbb{P}^1$ to every $(V, F, \hat F)$

• "Frobenius manifolds", C. Sabbah associates a $\mathbb{G}_m$-equivariant bundle $\xi(V, W, F)$ on $\mathbb{P}^1$ to every $(V, F^\bullet, W_\bullet)$

1. MHS and bundles on $\mathbb{P}^2$

Assume we are given $(V, F^\bullet, G^\bullet)$ - vector space with two exhaustive decreasing filtrations.

Let $R^2(V, F^\bullet, G^\bullet) = \langle u^{-p} v^{-q} w \mid w \in F^p \cap G^q \rangle \subset k[u,v]$

Then $(R^2(V, F^\bullet, G^\bullet))^\sim$ is a free $\mathcal{O}_{\mathbb{A}^2}$ module

Let $\xi_{\mathbb{A}^2}(V, F^\bullet, G^\bullet) := (R^2(V, F^\bullet, G^\bullet))^\sim$

Then $\xi_{\mathbb{A}^2}(V, F^\cdot, G^\cdot)$ is the subsheaf of $j_*(V \otimes \mathcal{O}_{\mathbb{G}_m^2})$ such that it is generated by $u^{-p} v^{-q} w$ with $w \in F^p \cap G^q$.

<u>Note</u>: • Here $j : \mathbb{G}_m^2 \hookrightarrow \mathbb{A}^2$.

• $\xi_{\mathbb{A}^2}(V, F, G)_{|(1,1)} = V$

• $\xi_{\mathbb{A}^2}(V, F, G)_{|(0,0)} = gr_F\, gr_G\, V$

• $\xi_{\mathbb{A}^2}(V, F, G)$ is $\mathbb{G}_m^2$ equivariant for the action covering the action

$$\mathbb{G}_m^2 \times \mathbb{A}^2 \to \mathbb{A}^2$$
$$((\lambda, \mu); (u, v)) \to (\lambda u, \mu v)$$

Thus we get functors

$$(V, F^\cdot, G^\cdot) \underset{\phi_I}{\overset{\phi_R}{\rightleftarrows}} \begin{pmatrix} \mathbb{G}_m^2\text{-equivariant} \\ \text{sheaves on } \mathbb{A}^2 \end{pmatrix}$$

Let now $(V, F_0^\cdot, F_1^\cdot, F_2^\cdot)$. Consider

$\mathbb{P}^2 = \mathrm{Proj}(k[u_0 : u_1 : u_2])$ $\quad k = \mathbb{C}$ for now

We have three affine opens on $\mathbb{P}^2$
U_0, U_1, U_2 and we can run Rees construction:

with (V, F_1, F_2) over $U_0 \Rightarrow \xi_{U_0}(V, F_1, F_2)$

with (V, F_0, F_2) over $U_1 \Rightarrow \xi_{U_1}(V, F_0, F_2)$

with (V, F_0, F_1) over $U_2 \Rightarrow \xi_{U_2}(V, F_0, F_1)$.

These glue together to a Π-equivariant sheaf

$$\xi_{\mathbb{P}^2}(V, F_0, F_1, F_2) \quad \text{on } \mathbb{P}^2$$

where $\Pi = \dfrac{G_m^3}{\Delta(G_m)}$ $\qquad \Delta : G_m \hookrightarrow G_m^3$

<u>Theorem:</u> the functors Φ_R, Φ_I induce equivalences of categories

$$\{(V, F_0, F_1, F_2), \text{any morph.}\} \underset{\Phi_I}{\overset{\Phi_R}{\rightleftarrows}} \left\{ \begin{array}{l} \Pi \text{ equivariant bundles} \\ + \Pi \text{ equivariant morphisms} \end{array} \right\}$$

$$\left\{ (V, F_0, F_1, F_2), \begin{array}{l} \text{strictly compatible} \\ \text{morphisms} \end{array} \right\} \underset{\Phi_I}{\overset{\Phi_R}{\rightleftarrows}} \left\{ \begin{array}{l} \Pi\text{-equivariant bundles} \\ \hline \Pi\text{-equivariant morphisms} \\ \text{whose cokernel has} \\ \text{codimension 2 torsion} \\ \text{only} \end{array} \right\}$$

Recall : $f : (V, F) \to (V', F')$ is strictly compatible $\Leftrightarrow f(F^p) = F'^p \cap f(V)$

<u>Def:</u> (V, F_0, F_1, F_2) are opposed if

$$gr^p_{F_0}\, gr^q_{F_1}\, gr^r_{F_2} \neq 0 \Rightarrow p+q+r=0.$$

<u>Theorem:</u>

$$\left\{ \begin{array}{l} (V, F_0, F_1, F_2) \text{ opposed, strictly compatible} \\ \text{morphisms} \end{array} \right\}$$

$$\Phi_R \big\downarrow \; \big\uparrow \Phi_I$$

$$\left\{ \begin{array}{l} \Pi\text{-equivariant bundles that are } \mu\text{-semi-stable} \\ \quad \text{of slope } \mu = 0 \\ \Pi\text{-equivariant morphisms whose cokernel} \\ \quad \text{has torsion in codimension 2} \end{array} \right\}$$

<u>Fact:</u> $c_1(\xi_{\mathbb{P}^2}(V, F_0, F_1, F_2)) = \sum_{p,q,r} (p+q+r) \dim_{\mathbb{C}} gr^p_{F_0}\, gr^q_{F_1}\, gr$

$\Rightarrow \mu((V, F_0, F_1, F_2)_{\text{opposed}}) = 0.$

To see that for (V, F_0, F_1, F_2) opposed $\Rightarrow \xi_{\mathbb{P}^2}(V, F_0, F_1, F_2)$ is μ-semistable consider

$$0 \to \mathcal{E} \longrightarrow \xi_{\mathbb{P}^2}(V, F_0, F_1, F_2)$$

and let $\mathcal{E}^{\vee\vee} \subset \xi_{\mathbb{P}^2}(V, F_0, F_1, F_2)$

Then $\Phi_I(\mathcal{E}^{\vee\vee})$ gives a trifiltered vector space with opposed filtrations.

But $\mu(\mathcal{E}) = \mu(\mathcal{E}^{\vee\vee}) = 0 \Rightarrow$
$\Phi_I(\mathcal{E}^{\vee\vee}) = (V, F_0, F_1, F_2)$.

This shows that $\xi_{\mathbb{P}^2}(V, F_0, F_1, F_2)$ is μ-semistable as a Π-equivariant bundle.

Conversely if $\xi_{\mathbb{P}^2}$ is semistable $\Rightarrow (V, F_0, F_1, F_2)$ are opposed.

<u>Note</u>: the above argument shows that the bundle will be μ-semistable as a Π-equivariant bundle. However if $\xi_{\mathbb{P}^2}$ is not semistable $\Rightarrow$ the maximal destabilizing subbundle of $\xi_{\mathbb{P}^2}$ will be Π-equivariant

and so will destabilize $\xi_{\mathbb{P}^2}$ as a

π-equivariant bundle.

Consider now $\tau : \mathbb{P}^2 \to \mathbb{P}^2$

$$(u_0 : u_1 : u_2) \to (\bar{u}_0 : \bar{u}_1 : \bar{u}_2)$$

For a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_{\mathbb{P}^2}$-modules define

$$\tau^* \mathcal{E}(U) := \mathcal{E}(\tau(U))$$

Thus $\tau^* \mathcal{E}$ is a sheaf of $\mathcal{O}_{\mathbb{P}^2}$ modules w.r.t. the multiplication

$$f \cdot e = \overline{\tau^*(f)} \cdot e \qquad \forall f \in \mathcal{O}_{\mathbb{P}^2}(U)$$
$$e \in \tau^* \mathcal{E}(U).$$

and τ^* induces an antilinear morphism

$$(*) \qquad \mathrm{Hom}(\mathcal{E}, \mathcal{E}') \xrightarrow{\tilde{\tau}^*} \mathrm{Hom}(\tau^* \mathcal{E}, \tau^* \mathcal{E}')$$

for any two $\mathcal{E}, \mathcal{E}'$ - coherent sheaves on $\mathbb{P}^2$.

Def: $\mathcal{E}$ is a τ-equivariant sheaf if there is an isomorphism

$$f : \mathcal{E} \xrightarrow{\sim} \tau^* \mathcal{E}$$

so that

$$\tau^*(f) \circ f = id_{\mathcal{E}}$$

where $\tau^*(f)$ is taken in the sense of $(*)$

Def: We will say that $\mathcal{E}$ is π^{τ} equivariant if $\mathcal{E}$ is both π and τ equivariant.

Theorem:

$$\left(\begin{array}{ll} (V, F_0, F_1, F_2) \text{ opposed} & \text{, strictly} \\ V = V_{\mathbb{R}} \otimes \mathbb{C} & \text{compatible} \\ \bar{F}_1 = F_2 & \text{morphisms} \end{array} \right)$$

$$\Phi_{\mathbb{R}} \Big\downarrow \Big\uparrow \Phi_{\mathcal{I}}$$

$$\left(\begin{array}{l} \pi^{\tau}\text{-equiv. bundles} \\ \mu\text{-semistable of slope } \mu = 0, \\ \text{---} \end{array} \right)$$

is an equivalence

Corollary: • The category of trifiltered opposed vector spaces with strict maps is abelian

• The category of real MHS are abelian.

Follows from the fact that the category of π-equivariant reflexive sheaves on a smooth manifold is abelian

To make the relationship to MHS explicit note that we can convert each

$$(H, W_\bullet, F^\bullet, \bar{F}^\bullet) \text{ into } (H, W^\bullet, F^\bullet, F^\bullet)$$

3. Discrete invariants of MHS

Given $(H, W, F, \bar{F})$ - real MHS $\Rightarrow$

$$c_2(\xi_{\mathbb{P}^2}(H, W, F, \bar{F})) = \frac{1}{2}\sum_{p,q}(p+q)^2(h^{p,q} - s^{p,q})$$

i.e. c_2 jumps when s^{pq} jumps.

Going back to our original example
$\Rightarrow$ when $D \subset (\mathrm{Re} = \frac{1}{2})$ $\Rightarrow$ $c_2 = 0$
when $D \not\subset (\mathrm{Re} \neq \frac{1}{2})$ $\Rightarrow$ $c_2 = 1$

In the first case the $\mathbb{R}$MHS is split
the second case the $\mathbb{R}$MHS is non-split

Now define

$$\alpha(H) := c_2(\xi_{\mathbb{P}^2}(H, W, F, \bar{F}))$$

Then we have

<u>Thm</u>
- $\alpha(H) \geq 0$
- $\alpha(H) = 0 \iff H$ is $\mathbb{R}$-split
- $\alpha(H \oplus H') = \alpha(H) + \alpha(H')$
- if $H \in \mathrm{Ext}^1_{MHS}(B, A)$

$$\alpha(H) \geq \alpha(A) + \alpha(B)$$

(<u>Remark</u>: It can happen that
$\alpha(H) = \alpha(A) + \alpha(B)$ but
H is not split.
For example $\alpha(H) = 0$ if $\mathrm{length}(H) \leq 1$.)

<u>Sketch of proof of the last statement:</u>

Given $0 \to A \to H \to B \to 0$ we get a ses

$$0 \to \xi(A) \to \xi(H) \to Q \to 0$$

a ses in the category of reflexive sheaves on $\mathbb{P}^2$ and $B = \Phi_I(Q^{\vee\vee})$

let $0 \to Q \to \xi(H) = Q^{\vee\vee} \to S_{(1,0,0)} \to 0$

$$\underset{c_2 \leq 0}{\uparrow}$$

$$\Rightarrow c_2(\xi(H)) + c_2(S_{(1,0,0)}) = c_2(\xi(A)) + c_2(\xi(B))$$

$\square$.