- The aim of this talk is to show how to extend from principal bundles to gerbes the notions of connections + associated curvatures.

- It's joint work with W. Messing (U. of Minnesota), math.AG/0106083
- It extends to the case of a gerbe with non abelian coefficient group the construction of J.L. Brylinski (Progr in Math 107) 1993.

- Our constructions rest upon a new description of differential forms and of the de Rham complex, described in (L.B, W.M "Combinatorial Differential forms", Adv. in Math 164, 263-282 (2001)), inspired by an analogous theory of A. Kock (Theory + Applic. of Categories $\underline{2}$, 69-89 (1996)) in the context of so-called Synthetic Differential Geometry.
- The first half of this talk will review this new interpretation of differential forms, which are then known as "Combinatorial" differential forms. The second half describes the application of this notion to connections on gerbes (after rapidly reviewing, in the present language, the well known theory of connections on principal bundles).

Def X an S-scheme defined (locally) by a k-algebra B.

$$\Omega^1_{X/S} := I/I^2 \quad \text{where } I \subset B\otimes B \text{ is the augmentation ideal}$$

defined by $0 \to I \to B\otimes B \xrightarrow{m} B \to 0$. In other words the sequence

$$0 \to I/I^2 \to \frac{B\otimes B}{I^2} \to B \to 0 \quad \text{is exact}$$

and

$$\Omega^1_{X/S} := \{ f : \Delta^1_{X/S} \to G_a \mid f_{|X} = 0 \}.$$

Here $\Delta^1_{X/S} := \mathrm{Spec}(B\otimes B/I^2)$ is the scheme of 1st order infinitesimally (principal parts) close points : $\Delta^1_{X/S} = \{(x,y) \in X^2 \mid x \sim y\}$. The more usual defn of $\Omega^1_{X/S}$ is as a function on $T_{X/S}$ linear on the fibers.

For $n > 1$, $\Omega^n_{X/S}$ is defined extrinsically as $\Omega^n_{X/S} := \Lambda^n_{O_X} \Omega^1_{X/S}$.

Let us set $\Delta^n_{X/S} := \{ (x_0, \dots, x_n) \mid x_i \sim x_j \quad \forall (i,j) \}$

For $X = \mathrm{Spec}(B)$ this infinitesimal prism is defined by :

$$\Delta^n_{X/S} = \frac{B^{\otimes n+1}}{\sum J_{i,j}^2}$$

where $J_{i,j}$ is the $(i+1, j+1)$th diagonal immersion of $X^n \hookrightarrow X^{n+1}$.

One then gets the following intrinsic description of $\Omega^n_{X/S}$

Thm1 (B, M) : When 2 is invertible in S,

$$\Omega^n_{X/S} \simeq \{ f : \Delta^n_{X/S} \to G_a \mid f(\dots, x_i, x_i, \dots) = 0 \ \forall i.$$

So the idea is to view a form as a volume form.

Cor: (A generalization of Kähler's definition to the $n > 1$ case):

$$\Omega^n_{X/S} \simeq \bigcap_i J_{i,i+1} \Big/ \sum J_y^2 \qquad \text{char} \neq 2$$

The properties of $\Omega^n_{X/S}$ with which we are familiar ~~more extended~~ can now be described in these combinatorial terms.

- $\forall \sigma \in S_{n+1}$, $\omega(x_{\sigma(0)}, \dots, x_{\sigma(n)}) = \operatorname{sign}(\sigma)\, \omega(x_0, \dots, x_n)$

- $$\begin{aligned} \Omega^i \times \Omega^j &\longrightarrow \Omega^{i+j} \\ (\omega, \eta) &\longmapsto \omega \wedge \eta \end{aligned}$$

where $(\omega \wedge \eta)(x_0, \dots, x_{i+j}) := \omega(x_0, \dots, x_i) \cdot \eta(x_i, \dots, x_{i+j})$.

- $$\begin{aligned} \Omega^n_{X/S} &\longrightarrow \Omega^{n+1}_{X/S} \\ \omega &\longmapsto d\omega \end{aligned}$$

where $(d\omega)(x_0, \dots, x_{n+1}) := \sum (-1)^i \omega(x_0, \dots, \hat{x}_i, \dots, x_{n+1})$

The de Rham complex is thus viewed as some infinitesimal form of the complex for Cech - Alexander cohomology.

The wonderful thing is that this approach is well suited to a description of differential forms with values in the Lie algebra of a group scheme.

Consider $\Psi^n_{X/S}(G) := \{ f : \Delta^n_{X/S} \longrightarrow G \mid f(\ldots x_i, y_i, \ldots) = 1 \quad \forall i \}$

<u>Theorem 2</u> When 2 not invertible on S

$$\Psi^n_{X/S}(G) \simeq \mathrm{Lie}\, G \otimes_{\mathcal{O}_S} \Omega^n_{X/S} .$$

<u>Properties</u> :
- Same behavior with respect to $\sigma \in S_{n+1}$
- It's an abelian group
- $\Psi^i_{X/S}(G) \otimes \Psi^j_{X/S}(G) \longrightarrow \Psi^{i+j}_{X/S}(G)$

$$\omega \qquad \eta \qquad \qquad \omega \wedge \eta$$

where now $(\omega \wedge \eta)(x_0, \ldots, x_{i+j}) := [\omega(x_0, \ldots, x_i), \eta(x_i, \ldots, x_{i+j})]$
is the classical pairing $\left(\mathrm{Lie}\, G \otimes \Omega^i\right) \times \left(\mathrm{Lie}\, G \otimes \Omega^j\right) \longrightarrow \mathrm{Lie}\, G \otimes \Omega^{i+j}$

$$(X \otimes \alpha \ , \ Y \otimes \beta) \longmapsto [X, Y] \otimes (\alpha \wedge \beta)$$

& satisfies the graded Jacobi identity.

- Finally, $\exists$ non abelian de Rham differentials :

$$\mathrm{Lie}\, G \otimes \Omega^1_{X/S} \longrightarrow \mathrm{Lie}\, G \otimes \Omega^2_{X/S}$$

$$\Psi^1_{X/S}(G) \xrightarrow{\ \partial^1\ } \Psi^2_{X/S}(G)$$

where $(\partial^1 \omega)(x_0, x_1, x_2) := \omega(x_0, x_1)\, \omega(x_1, x_2)\, \omega(x_2, x_0)$

$= d\omega + \omega \wedge \omega$ in traditional notation, or

$= d\omega + \frac{1}{2} [\omega, \omega]$

To the G-gerbe $\mathcal{P}$ is thus associated a monoidal stack $\mathcal{P}^{ad} := \mathcal{E}q(\mathcal{P}, \mathcal{P})$, the gauge monoidal stack of the G-gerbe $\mathcal{P}$.

Locally, objects in $\mathcal{P}^{ad}$ are described by morphisms from $TORS(G)$ to $TORS(G)$, & morphisms by natural transformations:

$$TORS(G) \underset{v}{\overset{u}{\Downarrow \theta}} TORS(G)$$

Even more locally, an object u in $\mathcal{P}^{ad}$ is described by an automorphism $u : G \to G$:

$$\begin{array}{ccc} TORS(G) & \xrightarrow{U} & TORS(G) \\ P & \longmapsto & u_*(P) := P \overset{G}{\wedge} G^{u} \end{array}$$

and a natural transformation $\theta : U \to V$ by a section g of G such that

$$v = i_g \circ u$$

where i denotes inner conjugation.

Last transparencies

Similarly, $\partial^2_\chi : \Psi^2_{X/S}(G) \longrightarrow \Psi^3_{X/S}(G)$

associated to $\chi \in \Psi^1_{X/S}(G)$:

$$(\partial^2_\chi \omega)(x_0, \dots, x_3) := \left({}^{\chi(x_0,x_1)}\omega(x_1,x_2,x_3)\right) \omega(x_0,x_1,x_3)\, \omega(x_0,x_2,x_3)^{-1}\, \omega(x_0,x_1,x_2)$$

The Bianchi identity states that $\partial^2_\chi(\partial^1 \chi) = 0$.

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in classical terms, $\partial^2_\chi \omega = d\omega + [\chi, \omega]$

Connection + curvature of a G-principal bundle: first three transparencies.

For us a G-gerbe $\mathcal{P}$ is a sheaf of categories locally of the form TORS(G) :

$U \subset X$: $\quad \mathcal{P}_{|U} \simeq \mathrm{TORS}(U; G)$.

since it's fibered in categories, self equivalences of $\mathcal{P}$ form a category (or stack), not a set as in the principal bundle case.

A connection ϵ on a G-principal bundle P is a morphism of G-principal bundles

$$\epsilon : p_1^* P \longrightarrow p_0^* P$$

above $\Delta^1_{X/S}$, in other words, for all $(x, y) \in \Delta^1_{X/S}$,

$$P_y \xrightarrow{\epsilon_{x,y}} P_x$$

with the further requirement that $\epsilon_{x,x} = 1_P$ for all $x \in X$.

The curvature κ of a connection ϵ is defined by the following commutative square above $\Delta^2_{X/S}$:

$$\begin{array}{ccc} p_2^* P & \xrightarrow{\epsilon_{12}} & p_1^* P \\ {\scriptstyle \epsilon_{02}} \downarrow & & \downarrow {\scriptstyle \epsilon_{01}} \\ p_0^* P & \xrightarrow[\kappa]{} & p_0^* P \end{array}$$

so that it is a morphism $\kappa : \Delta^2_{X/S} \longrightarrow P^{\mathrm{ad}}$ where

$$P^{\mathrm{ad}} := \mathrm{Aut}_G(P)$$

is the adjoint group bundle of P (so that κ is an element of the group $\mathrm{Lie}(P^{\mathrm{ad}}) \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$ of $\mathrm{Lie}(P^{\mathrm{ad}})$-valued 2-forms on X).

The Bianchi identity asserts that the diagram:

$$\begin{array}{ccc} p_0^* P & \xrightarrow{\kappa_{013}} & p_0^* P \\ {\scriptstyle \kappa_{023}}\big\downarrow & & \big\downarrow{\scriptstyle \mu_{01}(\kappa_{123})} \\ p_0^* P & \xrightarrow[\kappa_{012}]{} & p_0^* P \end{array}$$

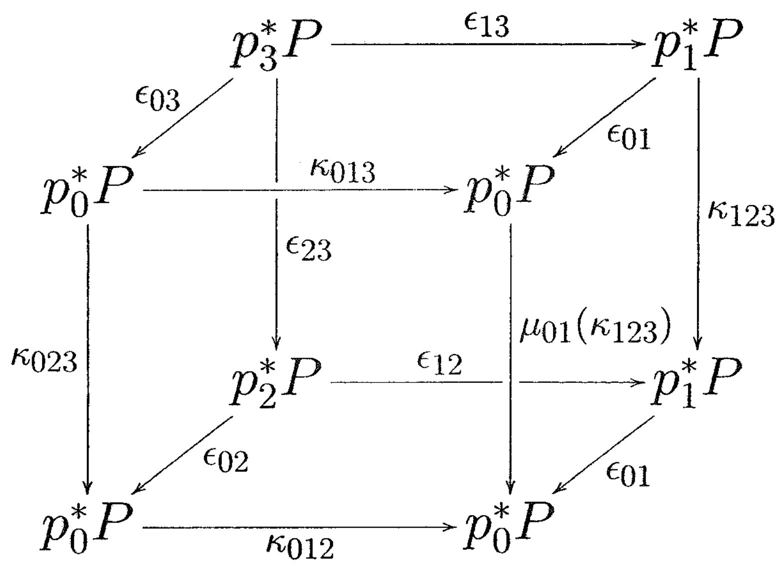
above $\Delta^3_{X/S}$ is commutative, in other words that the equation

$$\mu_{x,y}(\kappa_{y,z,u})\,\kappa_{x,y,u}\,\kappa^{-1}_{x,z,u}\,\kappa^{-1}_{x,y,z} = 1$$

is satisfied. This is equivalent to the classical Bianchi identity:

$$d_\mu(\kappa) := d\kappa + [\mu,\ \kappa] = 0.$$

Proof of the Bianchi identity:



The connection

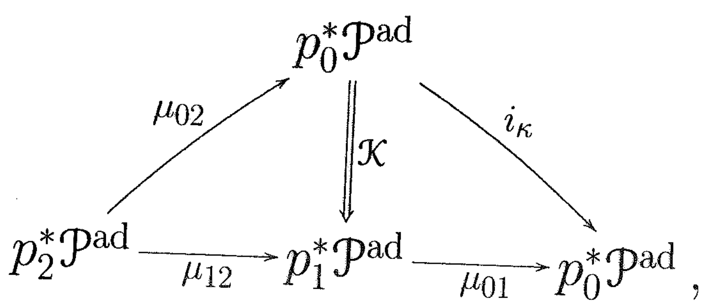
$$p_1^*\mathcal{P} \xrightarrow{\ \epsilon\ } p_0^*\mathcal{P}$$

determines a connection on the gauge stack $\mathcal{P}^{\mathrm{ad}}$:

$$\begin{array}{rccc} \mu = \epsilon^{\mathrm{ad}}: & p_1^*\mathcal{P}^{\mathrm{ad}} & \longrightarrow & p_0^*\mathcal{P}^{\mathrm{ad}} \\ & u & \longmapsto & \epsilon\, u\, \epsilon^{-1} \end{array}$$

Inner conjugation monoidal functor:

$$\begin{array}{ccc} \mathcal{P}^{\mathrm{ad}} & \xrightarrow{\ i\ } & \mathcal{E}q(\mathcal{P}^{\mathrm{ad}}, \mathcal{P}^{\mathrm{ad}}) \\ \kappa & \longmapsto & (u \mapsto \kappa\, u\, \kappa^{-1}) \end{array}$$



where $\mathcal{K} := i_K$.

In particular, the natural transformation $\mathcal{K}$ determines an arrow

$$\{\mathcal{K}, x\} : i_\kappa\ (\mu_{02})(x) \longrightarrow (\mu_{01})\ (\mu_{12})(x)$$

in $p_0^*\, \mathcal{P}^{\mathrm{ad}}$ for every object $x \in \mathrm{ob}(p_2^*\, \mathcal{P}^{\mathrm{ad}})$.

Connection ϵ on the gerbe $\mathcal{P}$:

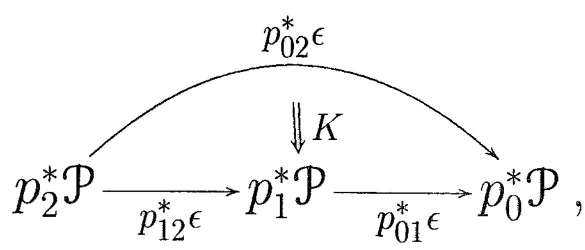
$$\epsilon : p_1^*\mathcal{P} \longrightarrow p_0^*\mathcal{P}$$

In other words, for all $(x, y) \in \Delta^1_{X/S}$, an arrow

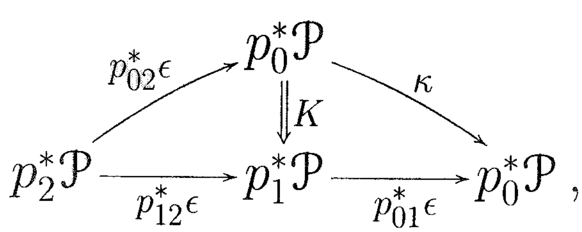
$$\mathcal{P}_y \xrightarrow{\epsilon_{x,y}} \mathcal{P}_x$$

with the further requirement that $\epsilon_{x,x} = 1_{\mathcal{P}_x}$ for all x.

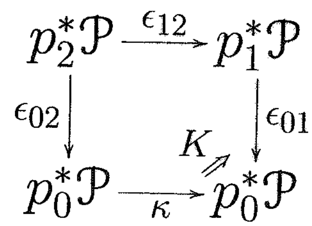
Naive 2-connection:



A (genuine) connective structure (κ, K) on the gerbe $\mathcal{P}$ consists of the **curving** K and the **fake curvature** κ:



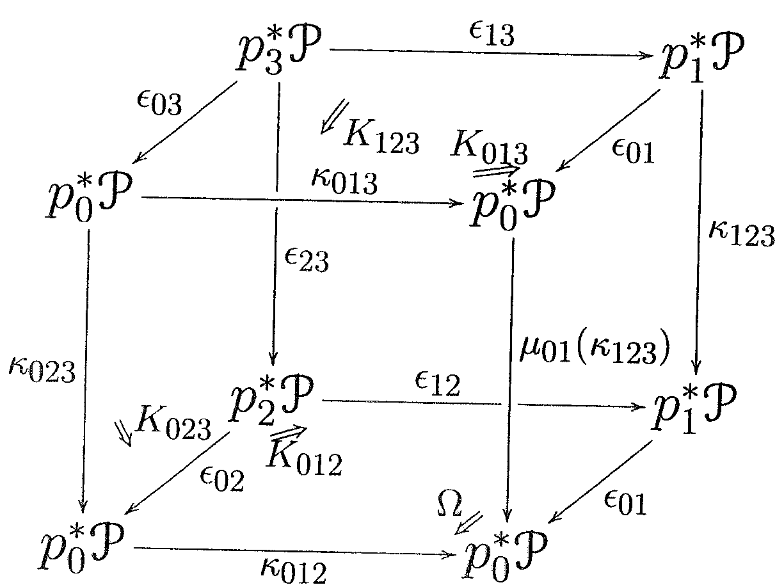
A curving triple (ϵ, κ, K) on a gerbe $\mathcal{P}$ consists of a **curving** K and a **fake curvature** κ:



The associated 3-curvature

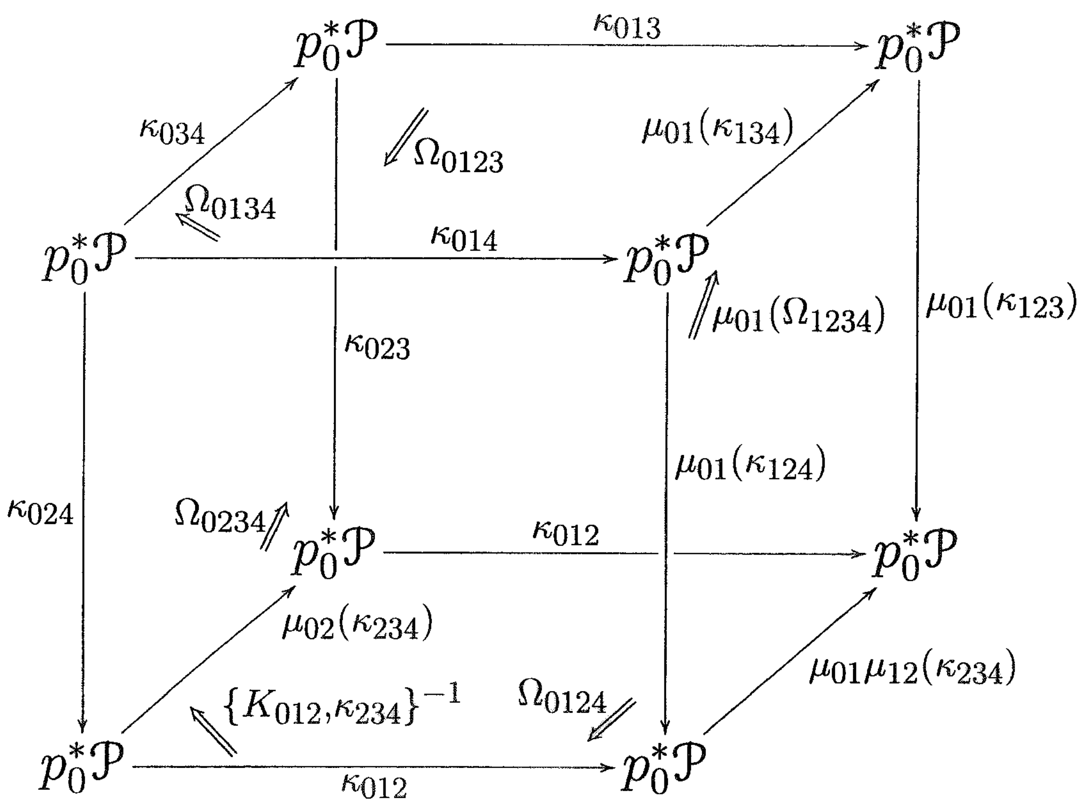
$$\Omega \in \mathrm{Ar}(\mathrm{Lie}(\mathcal{P}^{\mathrm{ad}})) \otimes_{\mathcal{O}_X} \Omega^3_{X/S}:$$

is the 2-arrow Ω defined as the front face of the commutative cube



We will call this cube the 3-curvature cube C.

The higher Bianchi identity asserts that the following cube above $\Delta^4_{X/S}$ is commutative:



The 1- and 2-arrows encountered above may locally be respectively described as Aut(G)- and G-valued differential forms, according to the following table:

ϵ	connection	Aut(G)	deg. 1	μ
K	curving	G	deg. 2	B
κ	fake curvature	Aut(G)	deg. 2	ν
Ω	3-curvature	G	deg. 3	ω

TABLE 1

Locally, the 3-curvature form is defined by

$$\omega = d_\mu B$$

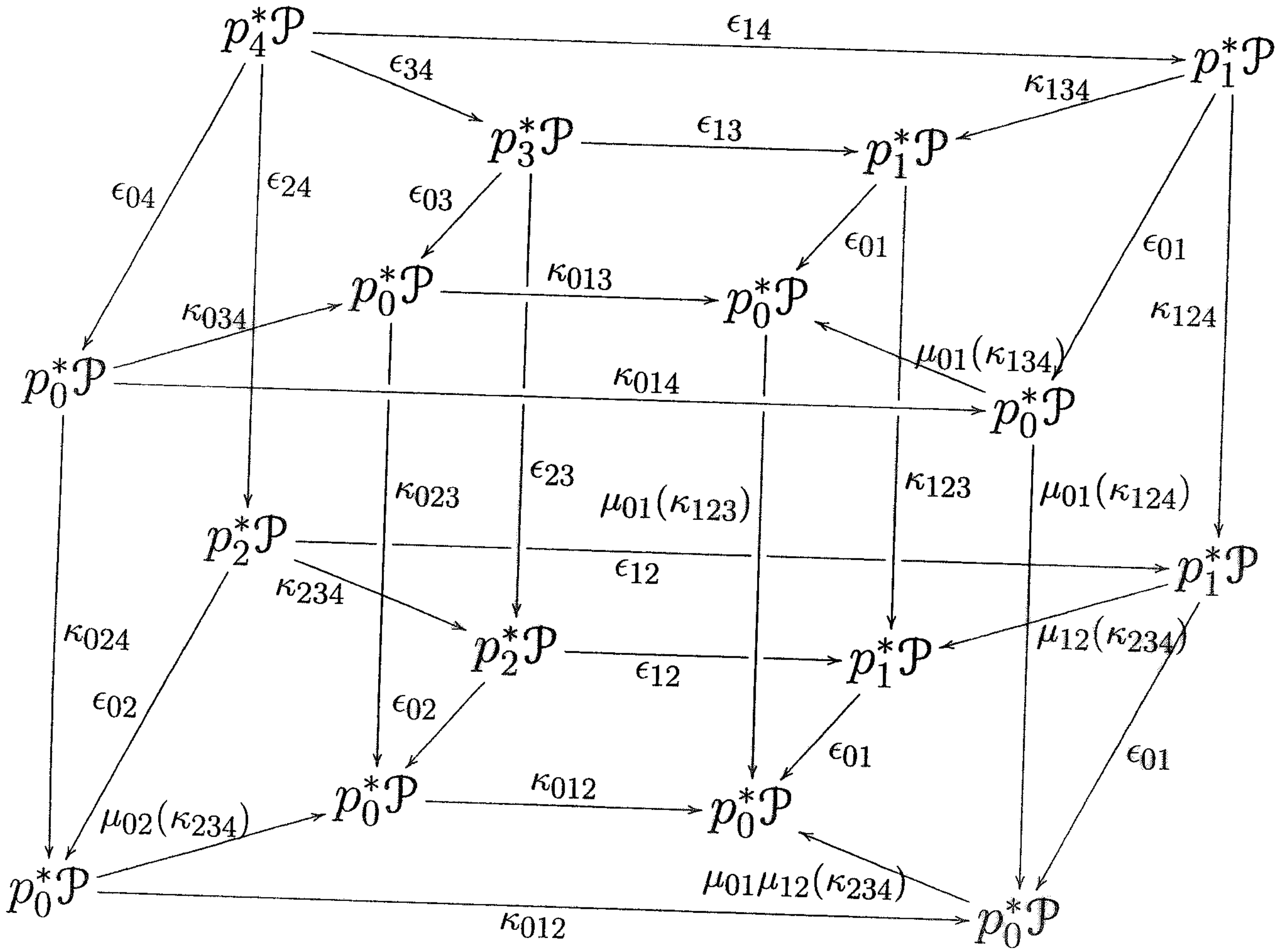
with $d_\mu B := dB + [\mu,\, B]$. Similarly, the higher Bianchi identity may be locally expressed as

$$d_\mu \omega := d\omega + [\mu,\, \omega] = [\nu,\, B]$$

where $[\ ,\ ]$ is the pairing between Aut(G)-valued and G-valued forms induced by the evaluation map

$$\mathrm{Aut}(G) \times G \longrightarrow G$$

Proof of the Bianchi identity:



The following table describes each of the eight cubes occuring in the previous hypercubical diagram:

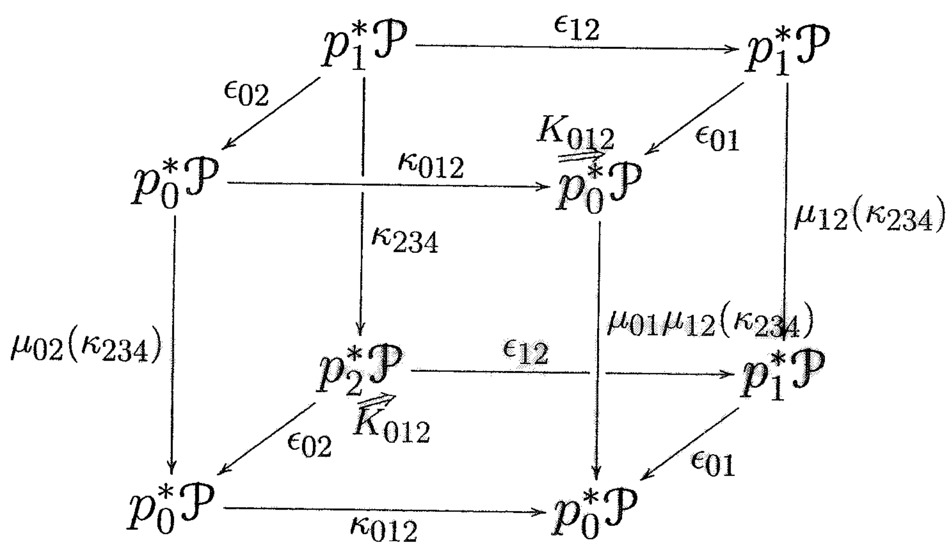
inner	left	right	top	bottom	front	back	outer
C_{0123}	C_{0234}	trivial	C_{0134}	$\{\, , \,\}$	Bianchi	C_{1234}	C_{0124}
	K_{023}	trivial	K_{013}	K_{012}	Ω_{0123}	K_{123}	

TABLE 2

The lower line in this table describes the face along which the corresponding cube is attached to the inner cube.

Here C is the cube defining the 3-curvature Ω.

The cube $\{\, , \,\}$ is the commutative cube



with trivial left, right and back faces, and which defines the 2-arrow $\{K_{012}, \kappa_{234}\}$ as its front face.