Sklyanin algebras and Hilbert schemes of points

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I'll begin the talk by telling you about some joint work with J.T. Stafford that describes Hilbert schemes of points on some noncommutative algebraic surfaces; then I'll explain the meaning behind the results and the motivation for studying these objects. Finally I'll try to explain how the moduli spaces that arise are related to moduli of Higgs bundles—note that there will be only "lower nonabelian Hodge theory" here.

To begin with, we'll need the following data.

E	a complex elliptic curve
$\mathcal{L}$	a deg. 3 line bundle on E
σ	a translation of E

Write Γ_σ for the graph of σ. Let

$$\mathcal{R} = H^0(E \times E, (\mathcal{L} \boxtimes \mathcal{L})(-\Gamma_\sigma)) \subset H^0(E, \mathcal{L})^{\otimes 2}.$$

Definition. The *3-dimensional Sklyanin algebra* associated to the above data is

$$S = S(E, \mathcal{L}, \sigma) = T^\bullet(H^0(E, \mathcal{L}))/(\mathcal{R}).$$

Some properties of this algebra are:

- When σ is the identity, $S \cong \mathbf{C}[x_1, x_2, x_3]$.

- In general, S is homogeneous coordinate ring of a noncommutative $\mathbf{P}^2$. This means that (a) S has global dimension 3 (i.e. every graded S-module has projective dimension at most 3), (b) S has finite GK-dimension (essentially, the size of the graded pieces of S grows polynomially in the grading) and in fact has the same Hilbert series as $\mathbf{C}[x_1, x_2, x_3]$, and (c) S is *Gorenstein* in the sense that $\operatorname{Ext}^q_S(\mathbf{C}, S) = 0$ if $q \neq 3$ and is $\mathbf{C}$ when $q = 3$.

- There is a $g \in S_3$ "corresponding to E," in the sense that $\operatorname{Proj}(S/(g))$ "is" E (to be precise, this means that the category of coherent sheaves on $\operatorname{Proj}(S/(g))$, which will be explained in a moment, is equivalent to $\operatorname{coh}(E)$), and $\operatorname{Spec}(S[g^{-1}]_0)$ should be thought of as a (noncommutative) $\mathbf{P}^2_S \setminus E$.

Remark. To be precise, it's best not to talk about any kind of space $\operatorname{Proj}(S)$ or $\operatorname{Spec}(S[g^{-1}]_0)$. Instead, one just imagines that such things exist and that this underlies the things which we can actually talk about, which are the categories of coherent sheaves on $\operatorname{Proj}(S)$ (the category of finitely-generated graded S-modules up to an equivalence relation which is essentially "two modules give the same coherent sheaf if they are isomorphic in large graded degree"—see a paper by Artin–Zhang or a survey paper by Stafford–van den Bergh for details) or of coherent sheaves on $\operatorname{Spec}(S[g^{-1}]_0)$ (just $S[g^{-1}]_0$-modules).

Fix an integer $n \geq 0$.

Let $\mathcal{M}(n, \sigma)$ denote the moduli space of stable filtered vector bundles $F_1 \subset F_2 \subset F_3$ on E for which

- $F_1 \cong (\mathcal{L}^*)^n$
- $F_3/F_2 \cong ((\sigma^{-1})^*\mathcal{L})^n$
- $F_3 \cong \mathcal{O}_E^{2n+1}$

Stability condition: for every (3-step) filtered subbundle $F'_\bullet \subset F_\bullet$, one has

$$2\deg(F'_3) + 3[\operatorname{rank}(F'_1) - \operatorname{rank}(F'_3/F'_2)] < 0.$$

Theorem. [Nevins–Stafford]

1. $\mathcal{M}(n,\sigma)$ is a nonsingular, connected, quasiprojective holomorphic symplectic variety of dimension $2n$.

2. When σ is the identity automorphism of E, $\mathcal{M}(n,\sigma)$ is isomorphic to $(\mathbf{P}^2 \setminus E)^{[n]}$ (Hilbert scheme of n points).

3. When σ has infinite order, the disjoint union

$$\coprod_{s\in\mathbf{Z}/3}\ \coprod_{n\geq 0} \mathcal{M}(n,\sigma)$$

classifies the rank 1 torsion-free coherent sheaves on $\mathrm{Spec}(S[g^{-1}]_0)$ (the noncommutative $\mathbf{P}^2_S \setminus E$).

Remarks.

a. Our study of the moduli space makes use of the "dimensional reduction" point of view in the spirit of Garcia-Prada, Bradlow... The symplectic pairing turns out to have already been constructed (in a more general setting) by Feigin–Odesskii.

b. Adding 1 to s corresponds to twisting by line bundle $\mathcal{O}(1)|_{\mathbf{P}^2\setminus E}$. So in some sense one still sees $\mathrm{Pic}(\mathbf{P}^2 \setminus E)$ even though there are many more line bundles now.

c. The different components seem to "fuse" from the moduli point of view in the case $|\sigma| = \infty$—that is, there appears to be only one component of $\mathrm{Pic}(\mathbf{P}^2_S \setminus E)$ for each value of s [note: I'm not absolutely confident of this statement, we haven't checked carefully], which are only separated by choosing extensions of line bundles on $\mathbf{P}^2_S \setminus E$ across E.

d. The family of moduli spaces $\mathcal{M}(n,\sigma)$ as σ varies (determined by a point of E) is flat over E.

e. I should be a little more careful in the statement of part 1 of the theorem. The statement is correct provided $\mathcal{M}(n,\sigma)$ is nonempty—which is true for generic σ as well as whenever σ has infinite order or is the identity map—but it doesn't seem to be obvious whether $\mathcal{M}(n,\sigma)$ is always nonempty.

f. One should think of $\mathcal{M}(n,\sigma)$ as the moduli space of line bundles on $\mathbf{P}^2_S \setminus E$ "with second chern class n," i.e. as a kind of analog of the Hilbert scheme of points on the commutative surface.

Antecedents: the affine plane

(1) Nakajima gave a concrete description of the hyperkähler metric on $(\mathbf{A}^2)^{[n]}$ via hyperkähler reduction. This motivates:

Question. Is there a geometric interpretation for the rotated complex structures on the Hilbert schemes?

I'm going to recount a (possibly revisionist) history that shows how one arrives at the study of $\mathcal{M}(n,\sigma)$:

(2) Cannings–Holland classified rank 1 torsion-free sheaves on a noncommutative $\mathbf{A}^2$ (more precisely, they classified rank 1 torsion-free modules over the first Weyl algebra, i.e. the ring of algebraic differential operators on the affine line) by points of an infinite-dimensional Grassmannian Gr^{ad}.

(3) G. Wilson showed that Gr^{ad} may be decomposed as a union of subspaces $\mathcal{C}_n$, each of which is a hyperkähler rotation of $(\mathbf{A}^2)^{[n]}$.

(4) Later Berest–Wilson showed how to directly realize the hyperkähler rotations $\mathcal{C}_n$ of $(\mathbf{A}^2)^{[n]}$ as moduli spaces of some torsion-free sheaves on the noncommutative $\mathbf{A}^2$ that was studied by Cannings and Holland.

(5) Since then there has been further work by Kapustin–Kuznetsov–Orlov on analogs of these spaces for higher rank sheaves, and work by Crawley-Boevey–Holland, Baranovsky–Ginzburg–Kuznetsov on noncommutative versions of the minimal resolutions of Kleinian singularities.

This work raises the following question (to which our theorem is an answer).

Question. Is this phenomenon (that deformations of the complex structure on the Hilbert scheme are explained by noncommutative deformations of the surface) a "linear" phenomenon? That is, does it only occur for the plane and its quotients?

Techniques

The techniques we use are similar in spirit to those used in other known cases, although of course there are some significant differences that arise because the geometry of $\mathbf{P}^2 \setminus E$ is different. In outline:

1. We prove that each torsion-free rank 1 sheaf on $\mathbf{P}^2_S \setminus E$ can be extended to a line bundle on $\mathbf{P}^2_S$ that is unique if one normalizes the degree of the restriction to E.

2. We prove that this induces a bijection between isomorphism classes of rank 1 torsion-free sheaves on $\mathbf{P}^2_S \setminus E$ and isomorphism classes of line bundles on $\mathbf{P}^2_S$ with given normalization.

3. We prove that the moduli space for such line bundles on $\mathbf{P}^2_S$ is isomorphic to a moduli space of monads, which is itself isomorphic to $\mathcal{M}(n, \sigma)$.

Moduli of Higgs bundles

Here I'll sketch some work in progress. It may seem somewhat strange (the conclusion will be that the moduli space $\mathcal{M}(n, \sigma)$ has an expression in terms of some rather more exotic objects) but the idea behind it is just to see whether one can find a way of thinking about $\mathcal{M}(n, \sigma)$ that gives it more of the structures that may be found in the work of Berest–Wilson, Nakajima, Baranovsky–Ginzburg–Kuznetsov, Kapustin–Kuznetsov–Orlov, etc.

Consider the surface $E \times \mathbf{P}^1$. Fix a C^∞ complex vector bundle V on $E \times \mathbf{P}^1$. One can think of this as a family of C^∞ vector bundles on E parametrized by $\mathbf{P}^1$; we will write the restriction to $E \times \{z\}$ as V_z, and will also write B_V when we want to think of V as a bundle living on $\mathbf{P}^1$—that is, B_V is that sheaf, the sections of which over an open subset $U \subset \mathbf{P}^1$ consist of C^∞ sections of $V|_{E \times U}$.

Let L_V denote the bundle of (infinite-dimensional) Lie algebras on $\mathbf{P}^1$ for which the fiber over $z \in \mathbf{P}^1$ is the space of C^∞ sections of $\mathrm{End}(V_z)$ over $E \times \{z\}$ (of course, I can also describe this sheaf-theoretically as I did for B_V). This has a bundle of central extensions $\widehat{L}_V$, which is roughly the central extension given by the cocycle

$$\Omega(X,Y) = \int_E \eta \wedge \langle X, \overline{\partial}_t Y\rangle$$

where η is a nonzero holomorphic 1-form on E and $\langle\, ,\, \rangle$ is the trace form (to formulate the cocycle properly in general replace the symbol $\overline{\partial}_t$ by any partial $\overline{\partial}$-operator in the E direction, for example by choosing a $\overline{\partial}$-operator on all of V and pulling back to T^*_E).

It makes sense to talk about operators on V that, locally on $E \times \mathbf{P}^1$ with coordinates t on E and z on $\mathbf{P}^1$, look like $\lambda \cdot d\bar{t} \cdot \partial/\partial\bar{t} + A \cdot d\bar{t}$, where A is a matrix of endomorphisms of V with coefficients that are C^∞ functions on $E \times \mathbf{P}^1$ (and $\lambda \in \mathbf{C}$). Let $\widehat{L}^*_V$ denote the bundle over $\mathbf{P}^1$ whose sections are such operators. This bundle is essentially the smooth part of the dual of $\widehat{L}_V$, so we can think of $\widehat{L}^*_V$ with the action of the group of automorphisms of V (also thought of as a bundle of groups over $\mathbf{P}^1$) as the bundle of coadjoint representations of a kind of centrally extended relative gauge group (associated to $\widehat{L}_V$).

By a $\overline{\partial}$-operator $\overline{\partial}_B$ on B_V I'll mean an operator

$$\overline{\partial}_B : B_V \longrightarrow B_V \otimes T^*_{\mathbf{P}^1}$$

on $\mathbf{P}^1$ that can be represented locally on $\mathbf{P}^1$ in the form $d\bar{z} \cdot \partial/\partial\bar{z} + B \cdot d\bar{z}$, where B is a section of $\widehat{L}_V$ that is C^∞ when thought of as a section of $\mathrm{End}(V)$ over $E \times \mathbf{P}^1$. A C^∞ *meromorphic Higgs field* Φ on $\mathbf{P}^1$ will be a section of $K_{\mathbf{P}^1}(D) \otimes \widehat{L}^*_V$ for some effective divisor D on $\mathbf{P}^1$, with the same kind of condition that the section be C^∞.

The link between filtered vector bundles on E and Higgs bundles on $\mathbf{P}^1$ is then the following correspondence.

Correspondence. Points of $\mathcal{M}(n,\sigma)$ correspond to meromorphic Higgs pairs $(\overline{\partial}_B, \Phi)$, up to isomorphism, on a certain C^∞ bundle over $E \times \mathbf{P}^1$. The relevant meromorphic Higgs pairs have poles at 0 and ∞ for which the residues lie in fixed coadjoint orbits in $\widehat{L}^*_V$; the relevant Higgs pairs are, moreover, $\mathbf{C}^*$-equivariant for the natural action on $\mathbf{P}^1$.

I won't state a precise theorem about isomorphisms of moduli today because some more checking of details needs to be done. This point of view was inspired by papers of Garland–Manton and Etingof–I. Frenkel.