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p-adic Hodge theory - part II

A_{cr} is a ring with: a $Gal(\bar{K}/K)$ action, Frobenius automorphism Φ, a filtration $F^i A_{cr}$ $i \geq 0$ where

$$F^i A_{cr} = (F^1 A)^{[i]} \quad \text{(divided powers ideal generated by } \frac{x^j}{j!}, j \geq i, x \in F^1)$$

Note: The filtration is Galois stable but not Φ-stable.

Furthermore:
- $A_{cr}/F^1 = \hat{V}$ (the p-adic completion)
- $\exists$ $\beta: \mathbb{Z}_p(1) \to F^1 A_{cr}$ which is Galois invariant and Φ linear i.e. $(\beta \circ \Phi = p\beta)$
- if t is the image of the generator in $\mathbb{Z}_p(1)$ under β, then

$$\begin{array}{ccc} F^i A_{cr}/F^{i+1} A_{cr} & \cong & \hat{V} \\ \cup & & \uparrow \\ \bar{V} \cdot \left(\frac{t^i}{i!}\right) & \cong & \hat{V}(i) \end{array}$$

(not Galois invariant iso)

(Galois invariant iso

multiplication by $(\zeta_p - 1)^i$, p-adic valuation $\frac{i}{p-1}$

In other words: $gr_F^{\cdot} A$ is (up to torsion) a direct sum of all Tate twists of $\hat{\bar{V}}$.

The ring A_{cr} can be used for understanding the period map for abelian varieties or p-divisible groups:

If A/V - abelian variety

$$\rho \in T_p(A) \longleftrightarrow \{ x_n \in A[p^n](\bar{V}) \} \quad p x_n = x_{n-1}$$

then we can lift A to A_{cr}; we have a map

$$EA(A_{cr}) \twoheadrightarrow A(\hat{\bar{V}})$$

whose kernel is isomorphic via the log map to

$$\mathrm{Ker}\,(\mathrm{Lie}(EA) \otimes A_{cr} \to \mathrm{Lie}(A) \otimes \hat{\bar{V}})$$

(this is not quite true if $p = 2$)

Then we can lift $\{x_n\}$ to $\{y_n\} \subset EA(A_{cr})$ and then to

$$\tilde{x}_n = \lim_{m \to \infty} p^m y_{n+m} \quad \text{- unique lift}$$

since

$$\begin{array}{ccccc} \mathbb{Q}_p & \longrightarrow & EA(A_{cr}) & & \\ \cup\!\text{I} & & \cup\!\text{I} & & \\ \mathbb{Z}_p & \longrightarrow & \text{kernel} & \xrightarrow{\log} & F^0(\mathrm{Lie}\, EA \otimes A_{cr}) \end{array}$$

($\mathrm{Lie}\, EA$ has a filtration with degrees $0, -1$).

In this way we get a period map

$$T_p(A) \otimes A_{cr} \to F^0(H_{1,cr}(A) \otimes A_{cr})$$

If we do the same for the dual abelian variety $A^\vee$ and dualize, then we will get a map in the opposite direction and the composition is $id \otimes \beta$.

Define $B_{cr} = A_{cr}[\frac{1}{t}, \frac{1}{p}]$ $\Rightarrow$ we get an isomorphism

$$(*) \qquad T_p(A) \otimes B_{cr} \cong H_{1,cr}(A) \otimes B_{cr}.$$

which respects the Hodge filtration in the following sense:

define B_{DR} = F-completion of B_{cr}, then $K_0 \subset B_{cr}$, $K \subset B_{DR}$ and we can form

$$H_{1,cr}(A) \otimes_{K_0} B_{DR}$$

and via the Hyodo-Kato isomorphism we have

$$H_{1,cr}(A) \otimes_{K_0} B_{DR} \cong H_{DR}(A) \otimes_{A} B_{DR}$$

The rhs now has a Hodge filtration (the product filtration) and the iso $(*)$ respects this filtration.

Because of this compatibility we will have

$$T_p(A) \otimes \mathbb{Q}_p = (T_p(A) \otimes B_{cr})^{\Phi = 1} \cap$$
$$\cap F^0(T_p(A) \otimes B_{DR})$$

$$H_{1,cr}(A) \otimes \mathbb{Q}_p = (H_{1,cr}(A) \otimes B_{cr})^{Gal\text{-}inv}$$

$$H^1_{DR} = (H_{1,cr}(A) \otimes B_{DR})^{Gal\text{-}inv.}$$

i.e. each cohomology theory can be reconstructed from the other.

We have a general comparison theorem

<u>Thm</u> (Fontaine-Messing, Kato, Tsuji, Faltings)
If X/V - proper and smooth $\Rightarrow \exists$

$$H^i_{et}(X \otimes \bar{K}, \mathbb{Q}_p) \otimes B_{cr} \cong H^i_{cr}(X) \otimes_{K_0} B_{cr}$$
$$\subset H^i_{DR}(X) \otimes_K B_{DR}$$

respecting Gal action, Φ-action and filtrations.

Def: A crystalline Galois representation M consists of:

- a Galois representation M
- $\exists\ E_0$ - a k_0-vector space equipped with Φ on $E = E_0 \otimes_{k_0} K$
- a filtration on E
- an isomorphism

$$M \otimes_{\mathbb{Q}_p} B_{cr} \cong E_0 \otimes_{K_0} B_{cr} \subseteq E \otimes_K B_{dR}$$

respecting Gal, Φ and $F^{\cdot}$

Note: • the crystalline Galois representations form a $\otimes$ category closed under taking subobjects

• representations of Gal coming from geometry are crystalline.

Non-abelian theory: Let C be a smooth projective curve over V.

Consider

$$\pi_1(C \otimes_V \bar{K}) \twoheadrightarrow G$$

$\uparrow$ maximal unipotent quotient = unipotent hull of $\pi_1/\mathbb{Q}_p$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$.

Given $x \in C(V)$ - rational point $\Rightarrow$ can form

$$\pi_1(C \otimes K) \cong \pi_1(C \otimes \bar{K}) \rtimes \mathrm{Gal}(\bar{K}/K)$$

<u>Theorem</u> $\mathfrak{g}$ is a crystalline $\mathrm{Gal}(\bar{K}/K)$ representation associated to an analogue of $\mathfrak{g}$ in crystalline cohomology.

<u>Idea of proof:</u> • technical reduction: remove one point from C and work with connections with logarithmic poles. The advantage of doing that is that G becomes <u>free</u>.

• Consider the regular representation of G on (functions on G) $\cong U(\mathfrak{g})^{dual}$

This is a smooth $\mathbb{Q}_p$-sheaf $\mathbb{L} = \bigcup_n \mathbb{L}_n$ and

$$\mathbb{L}_0 = \mathbb{Q}_p$$

$$0 \to \mathbb{L}_{n-1} \to \mathbb{L}_n \to V^{\otimes n} \to 0$$

$$V = H^1(C^0 \otimes \mathbb{Q}_p, \mathbb{Q}_p)$$

once punctured C.

<u>Note</u>: these extensions reflect the fact that G is not canonically free $\Rightarrow$ $U(\mathfrak{g})$ is not canonically isomorphic to a tensor algebra

The class of the extension $\mathbb{L}_n$ is given by $\mathrm{id}_{V^{\otimes n}}$.

We have
$$H^0(C^0 \otimes \bar{K}, \mathbb{L}_n) = \mathbb{Q}_p$$
$$H^1(C^0 \otimes \bar{K}, \mathbb{L}_n) \cong H^1(C^0 \otimes \bar{K}, V^{\otimes n}) \cong V^{\otimes n+1}$$

Assuming that we have a rational point $y \in C^0(V)$ we can kill the automorphisms of the extension by prescribing

$$\mathbb{L}_n(y) = \mathbb{Q}_p \oplus (\mathbb{L}_n/\mathbb{Q}_p)(y)$$

for fibers at y.

This uniqueness $\Rightarrow$ we can descent $\mathbb{L}_n$ to a local system on $C^0 \otimes K$.

Next we can do the same on the crystalline side and get vector bundles with connections on C or crystalline sheaves $\mathcal{E}_n$ on $C \otimes_V k / V_0$

Now we can compare the $\mathbb{L}_n$'s and the crystalline sheaves $\mathcal{E}_n$. For this we need to use a version of B_{cr}: $B_{cr}(\mathcal{O}_C)$ ↑ instead of V

$\mathrm{Spec}(R) \subset C$ affine "small"
↓ étale
$\mathbb{A}^1_V \supseteq$ log-divisor at ∞ for $C^\circ \subset C$.

form $\overline{R}$ (analogue of $\overline{V}$) = integral closure of R in the maximal unramified (outside of the divisor at ∞) of $R[\frac{1}{p}]$ (analogue of K).

Now by using a Tate-like argument we get

$$R_\infty \subseteq R$$

obtained by adjoining μ_{p^∞}, $t^{\frac{1}{p^\infty}}$.

The ring R_∞ is accessible to computations and $R_\infty \subseteq \overline{R}$ is "essentially étale"

Using R_∞ we can define $A_{cr}(R)$ and $B_{cr}(R)$ and then show as before that

$$\mathbb{L}_n \otimes B_{cr}(R) \xrightarrow{\sim} \mathcal{E}_n \otimes B_{cr}(R)$$

$\square$.

Remark: The ring $A_{cr}(R)$ is constructed as follows:

- consider $\bar{R}/p\bar{R} \circlearrowleft \Phi$
- consider

$$\mathcal{R} = \varprojlim \left(\bar{R}/p\bar{R} \xleftarrow{\Phi} \bar{R}/p\bar{R} \xleftarrow{\Phi} \cdots \right)$$

then

Φ acts on $\mathcal{R}$ and becomes an isomorphism on $\mathcal{R}$

- form $W(\mathcal{R})$ - the Witt vectors on $\mathcal{R}$. $\Rightarrow$ $W(\mathcal{R})/pW(\mathcal{R}) \cong \mathcal{R}$

Now the ring $A_{inf} = W(\mathcal{R})$ maps into $\hat{R}$ indeed:

$$W_n(X_1, \dots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + p^2 X_2^{p^{n-2}} + \cdots + p^n X_n$$

with $X_i \in \mathcal{R}$ i.e. $X_i = (X_{i0}, X_{i1}, X_{i2}, \dots)$
$\Rightarrow$ can form

$$(X_0, \dots, X_n) \to X_{0n}^{p^n} + p X_{1n}^{p^{n-1}} + \cdots + p^n X_{nn} \mod p^{n+1}$$

and this gives $A_{inf} \to \hat{R}$

The kernel is generated by 1 element ξ and

A_{cr} = p-adically complete divided power hull of A_{inf} (i.e. adjoin $\xi^n/n!$ $\forall n$)

$\mathbb{Z}_p$-theory: consider $V_0[[t]] \to V$
$f(\pi) = 0$
$f(T) = T^e + (0 \bmod p)$

Form R_V - the divided power hull p-adically completed (adjoin $f(t)^n/n!$ or $T^{en}/n!$)

$$\begin{tikzcd} & X/V\text{-smooth} \Rightarrow H^i_{cr}(X/R_V) \arrow[dl, "t \mapsto 0"'] \arrow[dr, "t \mapsto \pi"] & \\ H^i_{cr}(X \otimes_V k / V_0) & & H^i_{DR}(X/V) \end{tikzcd}$$

$\Rightarrow$ get a comparison map

$$H^i_{cr}(X/R_V) \otimes_{R_V} A_{cr} \to H^i_{et}(X \otimes \bar{K}, \mathbb{Z}_p) \otimes A_{cr}$$

$\Rightarrow$ get an inverse up to $p^{\otimes d}$
d = relative dimension.

This allows one to recover H^i_{et} if $d < p-1$ (since $t \in F^1/F^2$ has valuation $\frac{1}{p-1}$)

Example: for finite flat group-schemes
or abelian varieties of $d = 1$
$\Rightarrow$ get a good theory when $p \geq 3$
This shows up in Wiles, Breuil approach
to Taniyama - Shimura conjecture