Bertrand Toen 04-04-2002

Higher Tannaka duality

We would like to understand the relationship

{homotopy theory} $\longleftrightarrow$ {higher categories}

(motivated by "Pursuing stacks" and conversations with M. Spitzweck).

More precisely we will to understand

{spaces X} $\longrightarrow$ {category of certain representations of X} = {category of sheaves on X}

$\Longleftarrow$ ?

Plan: (1) Model categories versus higher categories

(2) Galois theory

(3) Tannakian model.

1. Model categories versus higher categories

We will consider $\infty$-categories such that all $i$-arrows are invertible for $i > 1$.

There are several different convenient models one can imagine here.

- One may work with simplicially enriched categories ($\underline{Hom}$'s are simplicial sets)
- One may work with Segal categories (introduced by Tamsamani–Simpson) – special class of simplicial categories where morphisms can be composed only up to homotopy.

We will work with Segal categories but one may work with quasi-categories (Joyal) or with complete Segal spaces (Rezk).

Segal categories work as well as usual categories, for example:

(a) for all Segal categories $A, B$ $\exists$ $\underline{Hom}(A, B)$ = weak functors $A \to B$ which is also a Segal category

Also

$$\underline{Hom}(A \times B, C) \cong \underline{Hom}(A, \underline{Hom}(B, C))$$

(b) there are notions of equivalences, limits, adjunctions, etc. for Segal categories.

(c) there is a notion of $\otimes$-Segal categories ($\Gamma$-objects in Segal Categories) $A \times A \xrightarrow{\otimes} A$.

there is a notion of modules over a monoidal Segal category $M \times A \to M$

there is a notion of a tensored and cotensored monoidal Segal category

(d) If $A$ is a monoidal Segal category $\Rightarrow$ we can talk of monoids in $A$: $Comm(A)$ – the Segal category of commutative monoids in $A$.

<u>Very important construction</u> (Dwyer-Kan localization

If $C$ – category, $\mathcal{S} \subset C$ – subcategory $\Rightarrow$ can construct a

$L(C, \mathcal{S})$ = Segal category obtained by "formally inverting maps in $\mathcal{S}$"

$L(C, \mathcal{S})$ is universal for maps of $C$ to Segal categories so that $\mathcal{S}$ go to isomorphisms:

$$\begin{tikzcd}
\mathcal{S} \arrow[d] & \subset & C \arrow[d] \arrow[r] & L(C, \mathcal{S}) \\
equiv(D) & \subset & D \arrow[ur, "\exists!"'] & \\
 & & \text{Segal category} \arrow[u] &
\end{tikzcd}$$

<u>Remark</u>: • Any simplicial category is a Segal category ⟹ any usual category is a Segal category

• If $M$ is a model category, $W \subset M$ - the weak equivalences ⟹ can form
$$LM = L(M, W)$$

<u>Thm</u> (Dwyer-Kan) If $M$ is a model category ⟹ $LM \cong$ simplicially enriched category of fibrant + cofibrant objects in $M$.

<u>Def</u>: $LM$ will be called the "Segal homotopy category" of $M$

<u>Examples</u>: • $Top = LSSet \sim LTop$

• $(LC(k), \otimes^{L})$ is a monoidal segal category. Here $C(k)$ is the model category of complexes of modules over a ring $k$.

<u>2. Galois theory</u>: Generalization of Grothendieck's $\pi_1$ - theory.

Let $X$ be a CW complex
$\pi_0(X) = *$ $x \in X$

$SPr(X)$ = category of simplicial presheaves on $X$

$LSPr(X)$ =: the Segal category of stacks of SSets on $X$

Consider $Loc(X) \subset LSPr(X)$ - the full sub Segal category of locally constant objects in $LSPr(X)$.

Consider $\omega_x : Loc(X) \to Top$ - evaluation ~~at~~ $x \in X$ functor.

Note $\omega_x \in \underline{Hom}(Loc(X), Top)$
$\uparrow$
Segal category.

Consider
$Aut(\omega_x)$ = space of auto equivalences of $\omega_x$ in $\underline{Hom}(Loc(X), Top)$

Then one can check that $Aut(\omega_x)$ is a loop space $\Rightarrow$ can form $BAut(\omega_x)$ (= delooping or classifying space)

Then one has

<u>Thm (Toen)</u> $\exists$ a natural weak equivalence

$$X \sim B\,\mathrm{Aut}(\omega_X)$$

<u>Note:</u> This theorem generalizes the usual statement that

$$\pi_1(X,x) \cong \mathrm{Aut}(\omega_x^0)$$

where

$$\omega_x^0 : \left\{ \begin{array}{c} \text{Loc constant} \\ \text{sheaves of sets} \\ \text{over } X \end{array} \right\} \to \text{Sets}$$

The previous theorem suggests that the whole theory can be linearized:

## <u>(3) Tannaka theory</u>

<u>Idea:</u> Replace the Segal category Top (serving as coefficients above) with a linear analogue e.g. $LC(k)$.

Assume: $k$ - noetherian ring
$X$ - finite CW complex
$x \in X$, $\pi_0(X) = 0$.

Consider

$C(X,k)$ = the model category of presheaves of complexes / $k$ on $X$ (local quasi isomorphisms)

$$LC(X,k) \supseteq LPerf(X,k)$$

$LPerf(X,k)$ = the full Segal subcategory of perfect objects i.e. objects locally q.i. to complexes

$$0 \to M_1 \to M_2 \to \cdots \to M_k \to 0$$

with $M_i$ - projective module of finite type / $k$.

- $LPerf(X,k)$ is a monoidal Segal category and it is enriched over

$$LPerf(k) \xrightarrow{p^*} LPerf(X,k)$$

where $p : X \to *$.

- $LPerf(X,k)$ has finite limits and colimits (the analogues of cones in the derived category).

- $LPerf(X,k)$ is a stable (or triangulated) Segal category (this is what replaces the notion of an abelian category in the Segal world).

Remark: There are alternatives to this "stable Segal category" notion. Joyal has defined "stable quasicategories". Kontsevich has defined "triangulated $A_\infty$ cat." Leinster has defined "triangulated higher category".

- $LPerf(X,k)$ is a rigid monoidal category, i.e.

$$\underline{Hom}(A,B) \cong A^\vee \otimes B$$

The "evaluation at a point" gives again a fiber functor

$$\omega_x : LPerf(X,k) \to LPerf(k) \subset LC(k)$$

which is conservative and exact.

Now we would like to construct

$Aut^{\otimes}(\omega_x)$ = the Tannakian dual to $(LPerf(X,k), \omega_x)$

Motivation for looking at $\underline{Aut}^{\otimes}(\omega_X)$:

If $k = \mathbb{C}$ then the loop stack
$\Omega_\bullet((X \otimes \mathbb{C})^{sch}) \cong \underline{Aut}^{\otimes}(\omega_X)$
so we can use it to define variations of $(X \otimes \mathbb{C})^{sch}$, by e.g. taking $k = \mathbb{C}[\varepsilon]/\varepsilon$

We will understand $\underline{Aut}^{\otimes}(\omega_X)$ as a group scheme over the category of complexes.

Thus we need to be able to do algebraic geometry over a monoidal Segal category. (Worked out by Toen and Vezzosi)

Start with

$LAff/k$ = the Segal category of affine schemes over $LC(k)$
$:= (Comm(LC(k)))^{op}$

There is a natural flat topology on $LAff/k$

let

$Sh(LAff_{/k})_{flat}$ = segal category of sheaves or stacks on $(LAff_{/k})_{flat}$ i.e.

$$F : (LAff/k)^{opp} \to Top$$

+ descent condition.

There is an Yoneda embedding

$$LAff/k \xhookrightarrow{h} Sh(LAff/k)$$

Image $(h)$ = affine schemes (or stacks).

Now we can define $\underline{Aut}^{\otimes}(w_X)$ as

$$\underline{Aut}^{\otimes}(w_X) : LAff/k \to Gp(Top)$$

$\Vert$ group objects in Top

$\Vert 2$

$L\underbrace{SGp}_{\text{simplicial groups}}$

So that

$$\underline{Aut}^{\otimes}(w_X)(A) = \underline{Aut}^{\otimes}(w \otimes A)$$

where
$$w \otimes A : LPerf(X,k) \to LC(k) \xrightarrow{\otimes_k A} LC(A).$$

Def: • $(X \otimes k)^{sch} := B\,\underline{Aut}^{\otimes}(w_X)$

• $\{k \mapsto (X \otimes k)^{sch}\}_{\text{all } k}$ is called the schematic shape of X

Thm $\underline{Aut}^{\otimes}(w_X)$ is an affine group scheme over $LC(k)$

In particular $\mathcal{O}(\underline{Aut}^{\otimes}(w_X))$ is an weak Hopf algebra ~~$k(k)$~~)

Example: If $k$ is a field then
$$\mathcal{O}_k^{sch}(X) := \mathcal{O}(\underline{Aut}^{\otimes}(w_X))$$
can be identified with the ring of functions on certain simplicial pro-algebraic group $G^{alg}$:

Consider $G =$ loop space of $X$ (cofibrant simplicial group)

$G^{alg}$ = the completion of $G$
(= simplicial affine group scheme)

Then $\mathcal{O}(G^{alg})$ = cosimplicial Hopf
algebra $/k$.