PLAN

JOINT WORK W/ BERTRAND TOEN !

1. Motivations

2. Overview of the Theory
   - Stacks (and topoi) $\begin{cases} \text{- Stacks over simplicial cat. \& model cat.} \\ \text{- model toposes} \end{cases}$
   - Homotopy algebraic geometry

3. Applications (DG-schemes; étale K-theory of ring spectra)

1. Motivations.

A • It has been known for quite a long time (Deligne, Hakim) that one can "do algebraic geometry" over any well behaved symm. monoidal category $(\mathcal{C}, \otimes)$:

[e.g. Tannakian: see Del, $\mathbb{P}^1 \setminus$ trois points]

Affine schemes $=$ (commutative monoids in $\mathcal{C}, \otimes)^{op}$

Some natural Groth. top. exists always (e.g. flat top)

$\Rightarrow$ can define sheaves, stacks, ...

* Ex. $(\mathcal{C}, \otimes) = (\mathbb{Z}\text{-mod}, \otimes)$ $\rightsquigarrow$ usual alg. geometry!
* Rmk But no "homotopy input" here!

B. Derived Algebraic Geometry. (Kontsevich, Kapranov, Ciocan-Fontanine, Hinich, ...)

Many moduli spaces arising naturally in AG are not smooth (eg stack of v.bundles on X $\dim X > 1$) and this may be considered unnatural and/or unpleasant $\rightsquigarrow$ "Hidden smoothness philosophy": the usual moduli space is just the 0-th truncation of a true extended/derived moduli space which is actually smooth.

Non smoothness arises from truncation [eg. we see jumps in dim. of tg spaces because we are actually looking at the 0-th part

Current approach to the problem: DG-schemes/stacks ...

Idea: embeds (sch) into (DG-sch) and extend moduli problems/functors to (DG-sch); look for repr.tive obj → get a DG-scheme/stack: the "true" moduli space.

Now: DG-schemes ~~theory~~ may be viewed as algebraic geometry over the sym. mon. model cat. of complexes $(C(k); \otimes; q\text{-iso})$ in the sense that (comm) algebras in this cat. are exactly (c) dga which are (dual to) affine objects in this AG.

Here: homotopy info = "q iso of complexes"

C. "Brave new" algebraic geometry

Recent progress in stable htpy theory lead to the construction of several (equivalent) monoidal model category of spectra (S-modules EKMM; symm. spectra SHH; simplicial functors Ly) so that it is now possible to think of E_∞-ring spectra as commutative monoids in any of these categories. ("Brave new" rings Waldh.)

$\mathbb{Z}$	$\mathbb{S}$ (sphere spectrum)
$Ab = \mathbb{Z}$-mod	$\mathbb{S}$-modules
monoids in Ab	monoids in $\mathbb{S}$-mod $\leftrightarrow A_\infty$-ring spectra
comm. monoids in Ab	comm. monoids in $\mathbb{S}$-mod $\leftrightarrow E_\infty$-ring spectra
$\otimes_{\mathbb{Z}}$	$\wedge_S$ (smash prod.)

$\Rightarrow$ have a "brave new" algebra here (alg.K-theory, THH, THC, ~~...~~ $\mathbb{L}$, TAQ, ...)

<u>Also</u>: recent rich interplay between AG/AT (eg. Elliptic cohomology, TMF)

$\Rightarrow$ Seems natural to embed this "brave new algebra" into a "brave new alg. geom." over the category (e.g.) ($\mathbb{S}$-modules, $\wedge_S$) where affine obj. are (comm. monoids in S-mod $\equiv$ comm. S-algebras).

D. Higher Tannaka duality

E. Manin's idea of AG over the category of motives ("secondary quantized AG")

F. Topological modular forms (?)

~~Model~~ ~~seems~~ ~~it~~

So: need AG over e.g. any symm. monoidal model category $(M, \otimes, \text{equiv})$ taking into account both the purely "algebraic" part $(M, \otimes)$ and the "homotopical" part (M, equiv).

Rmk Actually ~~can~~ think of this ~~AG~~ as a first step ~~to~~ in the development of AG over monoidal ∞-categories (e.g. ~~monoidal~~ monoidal Segal cats)

2.1

STACKS OVER A SITE $(\mathcal{C}, \tau)$

- τ : Groth. topology on $\mathcal{C}$
- $SPr(\mathcal{C})_{proj} := Funct(\mathcal{C}^{opp}, SSets)$
  model category of prestacks on $\mathcal{C}$
  [proj. model structure]

[Rmk cof = retracts of transfinite composition of pushouts along $\partial\Delta^n \otimes h_x \to \Delta^n \otimes h_x$, $x \in \mathcal{C}$
gen triv. cof = $\Lambda^n_k \otimes h_x \to \Delta^n \otimes h_x$.]

- Yoneda : (usual) $\mathcal{C} \xrightarrow{h} Pr(\mathcal{C}) \xrightarrow{c} SPr(\mathcal{C})$

Thm (B, DHI) $\exists$ model structure on $SPr(\mathcal{C})$ $(SPr(\mathcal{C})_{loc})$ with equivalences π_*-equivalences and cofibr = projective cofibrations. This is Quillen eq.t to the Bousfield loc. of $SPr(\mathcal{C})_{proj}$ along
:= hypercoverings $\{hocolim_{\Delta} U_\bullet \to h_x \mid U_\bullet \to ch_x$ hypercov$\}$
$L^B(SPr(\mathcal{C})_{proj}; HHC(\tau))$

Cor. Fibrant obj. in $SPr(\mathcal{C})_{loc}$ are objectwise fibrant prestacks having descent w/respect to all hypercoverings:
$F(x) \xrightarrow{\sim} holim\, F(U_\bullet)$

STACKS OVER A MODEL SITE (M, σ)

- σ : model topology on M ⓐ
- $SPr(M) := Funct(M^{opp}, SSets)$
  [proj. model structure]

- $M^\wedge := L^B(SPr(M), h_{equiv})$:
  $h_{equiv} := \{ c_* Hom_M(-, x) \to c_* Hom(-, y) \mid x \to y \text{ equiv} \}$
  model category of prestacks on M.
  [$Ho(M^\wedge) \hookrightarrow Ho(SPr(M))$: full subc. of obj. F s.t. $F(x \to y)$ equiv. of SSets $\forall\ (x \to y) \in equiv(M)$]

- h-Yoneda : choose Γ_* : cofibrant res. functor : $M \to sM$ [e.g. M smc $\Rightarrow \Gamma_*(-) = \Delta^* \otimes (-)$]
  $\Rightarrow$ $\underline{h} : M \to M^\wedge : x \mapsto \underline{h}_x : y \mapsto Hom_M(\Gamma_* y, x)$ $(n \mapsto Hom(\Gamma_n y, x))$
  $\underline{h}$ preserves eq. between fibrant $\Rightarrow$ ⓑ
  $\exists\ R\underline{h} : Ho(M) \to Ho(M^\wedge)$ & it is f.faithfull
  [homotopy Yoneda]

"inverting hypercovs" in $M^\wedge$ $\equiv \pi_*^\sigma$ - local equivalences

More precisely:
Thm $L^B(M^\wedge; HHC(\sigma))$ has equivalences π_*^σ - equivalences

Def. $M^{\sim,\sigma} := L^B(M^\wedge; HHC(\sigma))$
model cat. of stacks on the model site (M, σ)

Cf $Ho(M^{\sim,\sigma}) \hookrightarrow Ho(SPr(M))$
fully faithfull w/ essential image consisting of F s.t.
- $F(x \to y) \in$ equiv of SSets
- F has descent w/respect to σ-hypercoverings

$\Longleftarrow$ $\longleftrightarrow$ comparison

STACKS OVER A SIMPLICIAL SITE $(T, \underline{\tau})$

- $\underline{\tau}$: Groth. topology on $Ho(T)$
- $SPr(T) := Funct_s(T^{opp}, SSets)$
  model category of prestacks on T
  [proj. model structure]

- Yoneda (Suppose T fibrant i.e. all $\underline{Hom}$'s are fibrant ssets)
  $\underline{h} : T \to SPr(T)$ fully faithful (as S-cat to $SPr(T)_s$)
  $x \mapsto \underline{Hom}_T(-, x)$
  $\Rightarrow Ho(\underline{h}) : Ho(T) \hookrightarrow Ho(SPr(T))$ fully-faithful
  $(T_0, \pi_0 \underline{Hom}(-,-))$

"Invert $\underline{\tau}$-hypercov. in $SPr(T)$"

▶ $F \in SPr(T)$, $x \in ob(T)$, $\pi_0^{pr}(F)(x) := \pi_0(F(x))$
$\pi_0^{pr}(-) : Ho(T) \to Set$ $\Rightarrow$
$\exists\ \pi_0^{\tau}(-)$: associated sheaf on $(Ho(T), \tau)$
▶ $(f : F \to G) \in Ho(SPr(T))$ is a τ-covering if $\pi_0^\tau(f)$ is epi of sheaves on $(Ho(T), \tau)$
▶ $F_* \to G_*$ in $sSPr(T)$ is a $\underline{\tau}$-hypercov. if $\forall n$ ⓒ
$F_n \to \mathbb{R}cosk_{n-1}(G_*)_n$ is a τ-cov. in $Ho(SPr(T))$;
▶ $HHC(\tau) := \{ hocolim_{\Delta} F_* \to \underline{h}_x \mid F_* \to \underline{h}_x,\ x \in ob(T)$ τ-hypercov. & $F_n \underset{\text{iso in } Ho(SPr(T))}{\simeq} \coprod h_{u_\alpha} \}$
{homotopy hypercovers}

"local equivalences"

▶ As for π_0 we may define sheaves π_i^τ on $(Ho(T), \tau)$ (with base points)
Def. $f : F \to G$ is a π_*^τ-equivalence (or local eq.) if f induces isos on all π_i^τ's.

- Thm. $L^B(SPr(T)_{proj}; HHC(\tau))$ has equivalences $\equiv \pi_*^\tau$-equivalences.

- Def. $SPr(T)_\tau := L^B(SPr(T); HHC)$
  model category of stacks on (T, τ)

Cor: $Ho(SPr(T)_\tau) \hookrightarrow Ho(SPr(T))$ fully faithful w/ essential image consisting of obj ⓓ & $\exists$ left adj. ass.

– Semimodel cats

Same as model cat but :

- trivial cofibrations ~~with~~ cofibrant domain have LLP w/ all fibrations

- Factorization

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "\sim"'] & & y \\ & x' \arrow[ur, two heads] & \end{tikzcd}$$

only for x cofibrant

* Model pre-topologies on M & Topologies on $Ho(M)$

Def A model pretopology on M is the data of $Cov_\sigma(x)$ of families of maps in $Ho(M)$ such that

(i) (iso) $y \underset{iso}{\xrightarrow{\sim}} x \Rightarrow \{y \to x\} \in Cov_\sigma(x)$

(ii) (composition) "

(iii) (h-base change) $\{y_i \to x\} \in Cov_\sigma(x) \Rightarrow \forall (x' \to x) \in Ho(M)$

$\{y_i \times^h_x x' \to x'\} \in Cov_\sigma(x')$.

model pretop on $M \Longrightarrow$ $\underline{\sigma}$: Groth. Top. on $Ho(M)$ by declaring that a sieve over $x \in Ho(M)$ is covering if it contains a σ-covering family

$\overline{\alpha}$: model pretop. on M $\Longleftarrow$ α ; Groth top. on $Ho(M)$

$\{y_i \to x\} \in Cov_{\overline{\alpha}}(M)$ ~~iff~~ iff the sieve it generates is an α-covering sieve

$\Rightarrow$ Get a bijection between (saturated model pretop. on M) and (Groth. top on $Ho(M)$)

$\{\{y_i \to x\} \in Cov$ & $\{y_\alpha \to x\} \supseteq \{y_i \to x\} \Rightarrow \{y_\alpha \to x\} \in Cov\}$

- Sieve $/x$: $R = \{y_\alpha \to x\}$ s.t. $\forall\ y \to y_\alpha \Rightarrow \{y \to y_\alpha \to x\} \in R$

$R \Leftrightarrow R \hookrightarrow Hom(-, x)$

- Sieve generated by a family $\{y_i \to x\}$ = all maps to x factoring thru ~~some~~ some $y_i \to x$

- trivial Top. = $\forall x\ \exists !$ covering sieve : the maximal one $Hom(-, x)$

-

$\left[\{y_\alpha \to x\} \supseteq \{y_i \to x\} \Rightarrow \text{Sieve}\langle y_\alpha \to x\rangle \supseteq \underbrace{\text{Sieve}\langle y_i \to x\rangle}_{\text{covg}} \Rightarrow \text{Sieve}\langle y_\alpha \to x\rangle \text{ is covering.} \right]$

— $\underline{\underline{\mathbb{R}\,\mathrm{cosk}_n}}$ $\qquad \Delta_{\le n} \overset{j_n}{\hookrightarrow} \Delta$

$$\begin{tikzcd} SPr(T)^{\Delta^{op}} \equiv sSPr(T) \arrow[r, "j_n^*"] & s_{\le n}SPr(T) \equiv SPr(T)^{\Delta^{op}_{\le n}} \arrow[l, bend left=20, "(R)\ j_{n*}"'] \arrow[l, bend left=50, "j_{n!}\ (L)"] \end{tikzcd}$$

$$\mathrm{cosk}_n(-) := j_{n*}\, j_n^*(-) \qquad \text{Rmk} \left(j_n^* \mathrm{cosk}_n = \underbrace{j_n^* j_{n*}}_{\mathrm{id}} j_n^* = j_n^* \right)$$

$$\left(sk_n(-) := j_{n!}\, j_n^*(-) \right)$$

- Cosk_n is right Quillen $sSPr(T) \longrightarrow sSPr(T)$ (Reedy model str.)

  $\Rightarrow \exists\ \mathbb{R}\,\mathrm{Cosk}_n : Ho(SPr(T)) \longrightarrow Ho(SPr(T))$.

- <u>Relative $\mathbb{R}\mathrm{cosk}_n$</u> of $(F_* \xrightarrow{f} G_*) \in sSPr(T)$

$$\begin{tikzcd} \mathbb{R}\mathrm{cosk}_n(F_*/G_*) \arrow[r] \arrow[d] \arrow[dr, phantom, "Ph"] & G_* \arrow[d] \\ \mathbb{R}\mathrm{cosk}_n F_* \arrow[r, "\mathbb{R}\mathrm{cosk}_n(f)"'] & \mathbb{R}\mathrm{cosk}_n G_* \end{tikzcd}$$

by adj' $(\overset{l}{j_n^*}, \overset{r}{j_{n*}})$

— Hyperdescent (~~another point of view~~)

If $H_* \longrightarrow \underline{h}_x$ hypercovering and

$H_n \simeq \coprod_{i \in J_n} \underline{h}_{u_i}$, the hyperdescent condition becomes:

$$\boxed{F(x) \xrightarrow{\sim} \operatorname{holim}_{n \in \Delta} \left(\prod_{i \in J_n} H(u_i) \right)}$$ iso in $Ho(SSets)$

or, equivalently

$$\boxed{F(x) = \mathbb{R}\underline{Hom}(\underline{h}_x, F) \longrightarrow \mathbb{R}\underline{Hom}(\overbrace{\operatorname{hocolim}_{\Delta} H_*}^{\substack{|H_*| \text{ realization} \\ \shortparallel}}, F)}$$

iso in $Ho(SSets)$

– Functoriality

(T, τ) $\quad$ (T', τ') $\quad$ s-sites

$$f : T \longrightarrow T' \quad \rightsquigarrow \quad SPr(T) \underset{f^* \ (R)}{\overset{f_! \ (L)}{\rightleftarrows}} SPr(T')$$

Quillen adj.

Def f is continuous if f^* preserves the subcategories of stacks.

$\Downarrow$

induced Quillen adjunction for stacks

$$SPr(T)_\tau \underset{f^*}{\overset{f_!}{\rightleftarrows}} SPr_{\tau'}(T')$$

If f is continuous, $H_* \to \underline{h}_x$ τ-hypercovering, ~~f is continuous~~,

$$f_!(\operatorname{hocolim}_\Delta H_*) \longrightarrow f_!(\underline{h}_x) \simeq \underline{h}_{f(x)}$$

is an equivalence in $SPr(T')_{\tau'}$.

– Stacks of Morphisms : $\mathbb{R}_\tau \underline{\mathcal{H}om}(F, G) \in Ho(SPr(T)_\tau)$

for $F, G \in Ho(SPr(T)_\tau)$. [Because $Ho(SPr(T)_\tau)$ is Cartesian closed (use local model structure)]

Topologies & Model Topoi

Def. (Rezk) A *model topos* is a model category which is zig-zag Quillen equivalent to a left exact Bousfield localization of ~~the~~ $SPr(T)$ for some $T \in S$-Cat.

[*left exact*: $SPr(T) \xrightarrow{Id} L^B(SPr(T); ?)$ $\perp$ Id preserves htpy fiber products]

Theorem (for usual Groth. sites)

$\mathcal{C}$ cat.

$\{\text{topologies on } \mathcal{C}\} \xleftrightarrow{1-1} \{\text{strictly full subcategories } \mathcal{P} \overset{i}{\hookrightarrow} Pr(\mathcal{C})$ such that i has a left exact left adjoint$\}$

preserve finite limits

Theorem (for S-Sites)

$T \in S$Cat.

$\{\text{Simplicial top. on } T\} \leftrightarrow \{$left exact Bousfield localizat. of $SPr(T)$ which are t-complete (i.e. in which truncated objects detect isos in $Ho(-)$)$\}$

$\tau \mapsto SPr(T)_\tau$

Cor A model cat M is a model topos IFF it is zig-zag equivalent to the model cat. of stacks $SPr(T)_\tau$ on some simplicial site (T, τ)

Rmk . To have a bij t-complete is necessary. Ex. $\exists$ usual Groth. sites (T, τ) such that L (SPr(T), Čech hypercovers) $\not\simeq SPr(T)_\tau$. But $L^B(SPr(T), \check{C})$ is a left exact Bousfield localization of SPr(T) and both gives back τ!

. There probably exist hypertopologies on a cat or sCat = ~~data~~ of a bunch of hypercovers satisfying conditions similar to those imposed on covering families of a (pre)topology. [As above every T hypertopology all hypercov ≠ hypertop of all bounded covers]. Reasonable that {hypertopologies on} $\leftrightarrow$ {left exact loc.

— Comparison model vs. simplicial sites & stacks over them

(M, σ) model site $\longrightarrow$ $(L^{DK}M, \underline{\sigma})$

Thm.

$$M^{\sim,\sigma} \underset{\sim}{\rightleftarrows} SPr_{\underline{\sigma}}(L^{DK}M) \qquad (*)$$

Quillen equivalence

In particular $M^{\sim,\sigma}$ is a model topos.

Moreover the Yonedas are compatible (suppose σ subcan.)

$$\begin{tikzcd}
Ho(M) \arrow[r, "\text{model Yoneda}"'] \arrow[d, "\wr"] & Ho(M^{\sim,\sigma}) \arrow[d, "\wr\,(*)"] \\
Ho(L^{DK}M) \arrow[r, "\text{simplicial Yoneda}"] & Ho(SPr_{\underline{\sigma}}(L^{DK}(M)))
\end{tikzcd}$$

– truncated objects.

(T,τ) S-site then $\forall n \geq 0$ $\exists$ model structure on $SPr(T)$ ~~called~~ denoted by $SPr(T)^{\leq n}_{\tau}$ with same cof. as $SPr(T)$ and equivalences given by $\pi^{\tau}_{\leq n}$-equivalences

Actually $SPr(T)^{\leq n}_{\tau} = L^{B}\left(SPr(T)_{\tau}, \{\partial\Delta^{i}\otimes h_x \to \Delta^{i}\otimes h_x \ \forall x, i > n\}_{*}\right)$

$$Ho(SPr(T)^{\leq n}_{\tau}) \overset{\mathbb{R}Id}{\hookrightarrow} Ho(SPr(T)_{\tau})$$

fully faithful w/ ess. image objects n-truncated simplicial presheaves $(F : \pi_i(F(x), -) = 0 \quad i > n)$

:

$$t_{\leq n} : Ho(SPr(T)_{\tau}) \xrightarrow{\mathbb{L}Id} Ho(SPr(T)^{\leq n}_{\tau}) \overset{\mathbb{R}Id}{\hookrightarrow} Ho(SPr(T)_{\tau})$$

universal pty $\quad \mathbb{R}\underline{Hom}(t_{\leq n}F, G) \simeq \mathbb{R}\underline{Hom}(F, G)$ (iso in $Ho(SSets)$)

$\forall$ n-truncated G.

→ $\pi_0 : SPr(T) \to Pr(Ho(T))$ induces an equivalence of categories $Ho(SPr^{\leq 0}_{\tau}(T)) \simeq Sh_{\tau}(Ho(T))$ ← usual sheaves of sets on $(Ho(T), \tau)$

$(\Rightarrow Sh_{\tau}(Ho(T)) \hookrightarrow Ho(SPr(T)_{\tau}))$

→ More generally $\exists$ Quillen equivalence $SPr(T)^{\leq n}_{\tau}$ and a certain model category of presheaves of n-groupoids on the $(n+1)$-category $tr_{\leq n}(T)$ [← obtained from T by applying the n-th fund. groupoid to $T(x,y)$

– Remarks on Tab 1:

- Can replace proj model str. with injective mod.

  $\Rightarrow$ get Quillen eq. + model cats everywhere.

- In Tab 1 can replace SSets (the target category) with any "good" simplicial model category N

  $\rightsquigarrow$ get stacks ~~with~~ with values in N.

  {inverting hyperc.}

  In many cases (eg "algebraic theories", $N = Sp^{\Sigma}$, simplicial groups..) one can prove analogous results ~~of~~ local model structures (like the theorems in Tab 1) OR one can take eg. Sp^{Σ} in $SPr_{\tau}(T)$: this two constructions coincides (in good cases) see [BeKe].

- descent for bounded hypercov. ($\equiv \exists n : F_{\cdot} \xrightarrow{\sim} \mathrm{Cosk}_n F_{\cdot}$)

  $\Longleftrightarrow$ descent for nerves of coverings (Čech descent)

  But descent for bounded hc $\not\Rightarrow$ descent for all hc. in general

- First step towards AG over monoidal ∞-categories (eg. monoid

2.11 ALGEBRAIC GEOMETRY

Base category : $(\mathbb{Z}\text{-mod}, \otimes) \equiv Ab$

Commutative monoids in Ab : (alg)

Affine objects over Ab : $(Aff) := (alg)^{op}$

$Pr(Aff) \longrightarrow SPr(Aff) =: Aff^\wedge$

presheaves of sets on Aff

simpl. model cat of prestacks on Aff

Yoneda embedding:

$h : Aff \longrightarrow Pr(Aff) \xhookrightarrow{const} Aff^\wedge$

$X \longmapsto Hom(-,X) \longmapsto const\ Hom(-,X)$

So: $Aff \hookrightarrow Ho(Aff^\wedge)$

Grothendieck top on Aff : τ (eg flat)

Site (Aff, τ)

τ-hypercovers $\Longrightarrow S_\tau \subseteq Arr(Aff^\wedge)$

$Shv(Aff, \tau)$

Shaves of sets on (Aff, τ)

$Aff^{\sim,\tau} := L^B(Aff^\wedge, S_\tau)$

model cat of stacks on (Aff,τ)

$Ho(Aff^{\sim,\tau})$: htpy category of stacks (full subcat of $Ho(Aff^\wedge)$ of objects with hyperdescent)

τ subcanonical : $Aff \hookrightarrow Shv(Aff,\tau) \hookrightarrow Ho(Aff^{\sim,\tau})$

geometric stacks : $Ho(Aff^{\sim,\tau})_{geom.}$

ALGEBRAIC GEOMETRY OVER A SMMC

Base category : $(\mathcal{C}, \otimes, equiv) \equiv \mathbb{S}$

($\exists$ natural notion of homotopy-comm. algebra)

E_∞-algebras in $\mathbb{S}$: (Alg) (semi model cat)

Affine objects over $\mathbb{S}$: $\mathbb{S}\text{-}Aff := (Alg)^{op}$

$(\mathbb{S}\text{-}Aff)^\wedge$

simpl. model cat of prestacks on $\mathbb{S}$

Yoneda embedding (homotopy Yoneda)

$R\underline{h} : Ho(Alg) \hookrightarrow Ho(\mathbb{S}\text{-}Aff^\wedge)$

model top. on $\mathbb{S}\text{-}Aff$: τ ; model site $(\mathbb{S}\text{-}Aff, \tau)$

homotopy τ-hypercovers $\Longrightarrow S^h_\tau \subseteq Arr(\mathbb{S}Aff^\wedge$

$(\mathbb{S}\text{-}Aff)^{\sim,\tau}$, model cat of stacks over the model site $(\mathbb{S}\text{-}Aff, \tau)$

$Ho((\mathbb{S}\text{-}Aff)^{\sim,\tau})$: homotopy cat. of stacks over the

(full subcat of $Ho(\mathbb{S}\text{-}Aff^\wedge)$ - - - of objects having hyperdescent)

τ subcanonical $\mathbb{R}h : Ho(\mathbb{S}Aff) \hookrightarrow Ho(\mathbb{S}\text{-}Aff^\sim$

geometric stacks : $Ho(Aff^{\sim,\tau})_{geom.}$

– Geometric stacks

(Simpson)

By induction: 0-geometric stacks are "affine"

0-covering family $\{f_i : F_i \to G\}$ → each F_i is 0-repr.ble (base change by an Aff is 0-)
→ $\forall$ 0-geom $H \to G$ htpy pullback is a τ-covering family

n-geometric stacks F: $F \simeq \underset{\Delta}{\mathrm{hocolim}}(X_\bullet) = |X_\bullet|$

($\amalg$ colim)

where $X_\bullet \in s(\$\text{-}\mathrm{Aff}^{\sim,\tau})$ is a Segal groupoid $\left(X_n \xrightarrow{\sim} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1\right.$

& (essentially) $\mathrm{Ho}(X_\bullet)$ is a groupoid object in $\mathrm{Ho}(\$\text{-}\mathrm{Aff}^{\sim,\tau})\Big)$

such that:

- $X_0 = \coprod (n-1)$-geometric stacks
- $\{X_1 \xrightarrow{d_0} X_0\}$ and $\{X_1 \xrightarrow{d_1} X_0\}$

  is a $(n-1)$-covering family

3. Applications

3.1 Comparison with Kapranov & Ciocan-Fontanine Dg-schemes

Here the base monoidal model category is $\mathbb{D} = C(k)$ complexes of k-modules (k any ring for now).

$\rightsquigarrow$ Have $C(k)\text{-}Aff := (E_\infty\text{-algebras in } C(k))^{op}$

This is a genuine model category (not only semi) [Hinich]

Any topology τ on $(Sch_{/k})$ (eg. flat, étale, ...) gives a strong τ-topology [Behrend] on $C(k)$-Aff denoted by $h\tau$ defined by:

$$\{A \xrightarrow{f_i} B_i\} \in Cov_{h\tau}(A) \text{ iff}$$

- $\{\mathrm{Spec}(H^0(B_i)) \longrightarrow \mathrm{Spec}\, H^0(A)\}$ is a τ-covering family (of schemes)
- $\forall i$, $H^*(A) \otimes_{H^0(A)} H^*(B_i) \underset{iso}{\simeq} H^*(B_i)$.

Other examples of topologies: (1) $Cov_{H\tau}(A)$: $\{H^*(A) \to H^*(B_i)\}$ τ-cover

(2) $Tor_{\geq 0}$: each $\mathbb{L}f_i^* : Ho(Mod_A) \xrightarrow{\otimes^{\mathbb{L}}_A B_i} Ho(Mod_B)$ preserves positive modules ($\equiv M$: $H^i(M) = 0$ $i < 0$

- $\{\mathbb{L}f_i^*\}$ is conservative. (covering $\circ$ Sur^j f_i cond)

$C(k)\text{-}\mathrm{Aff}^{\sim,h\tau}$: model category of stacks on $C(k)$ for the strong τ-topology.

Prop. τ coarser than the ffqc topology $\Rightarrow$ $h\tau$ is subcanonical. In this case Yoneda factors as:

$$Ho(C(k)\text{-}\mathrm{Aff}) \xrightarrow{\mathbb{R}\underline{h}} Ho(C(k)\text{-}\mathrm{Aff}^{\sim,h\tau})$$

DG-schemes: I will not recall what they are; just say that ~~th~~ same def.n can be given replacing sheaves of dga$_{\leq 0/k}$ with ~~sheaves~~ of E_∞-alg$_{/k}$ in ≤ 0 deg. The homotopy categories are the same if ~~k~~ field of char $=0$.
(All schemes will be sep & q.compact).

Theorem. There exists a functor $\Theta : Ho(DG\text{-}sch_{/k}^{E_\infty\text{-}Sch_{/k}}) \longrightarrow Ho(C(k)\text{-}\mathrm{Aff}$,
($\tau \leq$ ffqc) such that:

(1)
$$\begin{tikzcd} Ho(DG\text{-}Sch_{/k}) \arrow[rr, "\Theta"] & & Ho(C(k)\text{-}\mathrm{Aff}^{\sim,\tau}) \\ & Ho(C(k)\text{-}\mathrm{Aff})^{\leq 0} \arrow[ul, "Spec"] \arrow[ur, "(\text{yoneda})"', "\mathbb{R}\underline{h}"] & \end{tikzcd}$$

commutes;

(2) $\Theta(X)$ is a (1-) geometric stack $\forall$ DG-scheme X.

⌈ Idea of the construction (Čechiste):

$(X, \mathcal{A}^\cdot_X)$: DG-scheme (sep & q.c.) $\qquad \mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ finite affine open cover

$$A_n := \mathcal{A}^\cdot_X \Big(\coprod_{I^{n+1}} U_{i_0, \dots, i_n} \Big) \rightsquigarrow$$

$$\mathbb{R}h_{A_*} : \Delta^{op} \longrightarrow C(k)\text{-}\mathrm{Aff}^{\sim, \mathrm{ft}, \mathrm{qc}} : [n] \longmapsto \mathbb{R}\underline{h}_{A_n}$$

Def. $\Theta_{\mathcal{U}}(X, \mathcal{A}^\cdot_X) := \operatorname{hocolim}_{[n] \in \Delta^{op}} (\mathbb{R}\underline{h}_{A_*})$.

It is independent of $\mathcal{U}$. &, by construction, the diagram in (1) commutes. ⌋

Conj. Θ is fully-faithful.

Question: * Define "moduli problems/functors" (eg. curves, v.b.) on the category $(C(k)\text{-}\mathrm{Aff}^{\sim, \tau})$ of stacks over complexes $/k$?

* (weak) Representability?

3.2 Étale K-theory of comm. ring spectra.

I'll state everything ~~for~~ $\mathbb{S}$-modules $\mathbb{S}$-algebras ...
(EKMM) but any other good model ($\equiv$ monoidal)
of spectra will do (HSS: Sp^{Σ}, Lydakis: Simpl. functors)

- $Aff_{\mathbb{S}} := $ (commutative S-algebras $\equiv$ comm. monoids in S-modules$)^{op}$
(S sphere spectra)

- $A \in Aff_S \rightsquigarrow Mod_A$: A-modules.

- $Ho(Mod_A)$ is a closed symm. monoidal cat ($\otimes$ induced by the smash product $- \wedge_A =$)

$\rightsquigarrow Mod_A^{sd}$: strongly dualizable cofibrant objects in Mod_A (Waldhausen cat. w eq. & cof induced from Mod_A)

$\rightsquigarrow$ can organize: $K(-) : Aff_S \longrightarrow SSet$

$$\operatorname{Spec} A \longmapsto K(Mod_A^{sd}) =: \mathbb{K}(A).$$

(symbolically $\equiv$ A viewed as an object in the opposite cat)

$K(-) \in Aff_S^{\wedge}$ (prestack of algebraic K-theory on comm. S-algebras) [skip Rmks. (1) Can also define a spectra valued prestack using the whole Ω-spectrum of Waldhausen K-theory ...]

(2) Can also define K-theory 1-Segal stack of any stack $X \in Aff_S^{\sim,\tau}$ for any topology τ on Aff_S : replace Mod_A^{sol} with $Perf(X)$

[Toen's talk]

Étale topology on Aff_S : (ét):

$\{A \xrightarrow{f_i} B_i\}$ of comm. S-alg is in $Cov_{ét}(\operatorname{Spec} A)$ if it is a finite family and:

- [formally étale] $\forall i$ $TAQ^A(B_i; B_i)$ (Basterra Def.) is trivial / $\mathbb{L}_{B_i/A}$ trivial

  / $B_i \longrightarrow \underbrace{S^1 \otimes^{\mathbb{L}}_A B_i}_{THH(B_i)}$ is an iso in $Ho(Alg_A)$ ($\otimes^{\mathbb{L}}$ performed in Alg_A)

  / $Map_{Alg_A}(B_i, C)$ is a discrete simpl. set

  (étale)

- [finite presentation] $\forall$ filtered direct $C : J \to Alg_A$

$$\operatorname{hocolim}_J Map_{Alg_A}(B_i, C(j)) \xrightarrow{\sim} Map_{Alg_A}(B_i, \operatorname{hocolim}_J C(j))$$

equiv.ce of s.sets.

- [Faithful / surjectivity condition] the family $\{\mathbb{L}f_i^* : Ho(Mod_A) \xrightarrow{-\otimes^{\mathbb{L}}_A B_i} Ho(Mod_{B_i})\}$ is conservative

[**Lemma.** $f : A \to B$ ~~of~~ of connective comm. $\mathbb{S}$-algebras (of finite pr.). Then f is étale iff:

(i) $\pi_0(A) \longrightarrow \pi_0(B)$ étale

(ii) $\pi_*(B) \longleftarrow \pi_*(A) \otimes_{\pi_0(A)} \pi_0(B)$ is a ring iso.]

Def. $K_{\acute{e}t}$:= fibrant replacement of K in $\mathrm{Aff}_S^{\sim,\acute{e}t}$

R discrete comm. ring $\quad K_{\acute{e}t}(HR) \simeq K_{\acute{e}t}(R)$ (??) [Problem: Small étale sites $/R$ & $/HR$ are "equivalent" ?]

Ex. $\mathbb{K}o$ real connective top. K-theory spectrum

$\mathbb{K}u$ complex " " " "

$\mathbb{K}u^{hC_2} = \mathbb{K}o \quad \Rightarrow \quad K_{\acute{e}t}(\mathbb{K}u)^{hC_2} = K_{\acute{e}t}(\mathbb{K}o)$

?? $K_{\acute{e}t}(\mathbb{K}u)$: model of elliptic cohomology ??

Étale cohomology (w/ constant coeff.'s)

M: abelian grp. (eg $M = \mathbb{Z}/\ell$) $\rightsquigarrow K(M,n) \in \mathrm{Aff}_S^{\wedge}$ (const. + simpl. presh.)

$\rightsquigarrow$ Fibrant repl. $K(M,n)_{\acute{e}t}$ in $\mathrm{Aff}_S^{\sim,\acute{e}t} \rightsquigarrow A \in \mathrm{Alg}_S \mapsto H^n_{\acute{e}t}(A;M) := K(M,n)_{\acute{e}t}($

$H^*_{\acute{e}t}(HR;M) \simeq H^*_{\acute{e}t}(R;M)$? ; Sim. for stable ... $\to$ étale K-theory ??