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<u>On the equivalences of derived categories of coherent sheaves</u>

X - smooth projective varieties
to each X associate Coh(X) - the abelian category of coherent sheaves on X

More convenient - consider $D^b(X) = D^b(Coh(X))$
the derived category of coherent sheaves

$D^b(X)$ is the main object we would like to study. $D^b(X)$ is a triangulated additive category:

(a) $D^b(X)$ is endowed with a translation functor

$$[1] : D^b(X) \to D^b(X)$$

which is an additive autoequivalence

(b) $D^b(X)$ is endowed with a class of distinguished (exact) triangles

$$X \to Y \to Z \to X[1]$$

satisfying a collection of axioms (Verdier)

We want to study the following important questions:

Questions: (1) When do two different varieties have equivalent derived categories of coherent sheaves

(2) How can we describe all equivalences between $D^b(X)$ and $D^b(X')$?

Example (Mukai) Let A be an abelian variety and let $\hat{A}$ be the dual abelian variety : $\hat{A} = Pic^0(A)$

Note : $\hat{A}$ is an abelian variety of the same dimension and in general $\hat{A} \not\cong A$ (e.g. if A does not have a principal polarization)

We have $P \to A \times \hat{A}$ - Poincare line bundle

and projections

$$\begin{array}{ccc} A \times \hat{A} & \xrightarrow{q} & \hat{A} \\ {\scriptstyle p}\downarrow & & \\ A & & \end{array}$$

Thm (Mukai) The Fourier-Mukai functor

$$\Phi_P : D^b(A) \to D^b(\hat{A})$$
$$F \to Rq_*(P \otimes p^*(F))$$

is an equivalence of categories.

<u>Thm (Bondal-Orlov)</u> Let X be a smooth projective variety with either ample or anti-ample canonical sheaf.

If $D^b(X) \cong D^b(X')$ for some proper algebraic variety X' $\Rightarrow$ $X \cong X'$.

<u>Note</u>: The reason the previous theorem holds is because of the existence of a Serre functor

$$S : D^b(X) \to D^b(X)$$
$$F \to F \otimes \omega_X[n]$$

on every X.

The Serre functor can be defined completely in categorical terms: it is the unique-up-to-equivalence functor satisfying

$$\mathrm{Hom}(G, S(F)) \cong \mathrm{Hom}(F, G)^\vee.$$

$\forall$ F, G.

The existence of Serre functor makes $D^b(X)$ somewhat rigid and combined with vanishing coming from ampleness or anti-ampleness gives us enough rigidity to reconstruct X.

Thm (Bondal - Orlov) Let X be a smooth projective variety with either ample or anti-ample canonical bundle. Then

$$\mathrm{Aut}(D^b(X)) \cong \mathrm{Aut}(X) \ltimes (\mathrm{Pic}\, X \oplus \mathbb{Z})$$

Note: $\mathrm{Aut}(D^b(X)) \supseteq \mathrm{Aut}(X) \ltimes (\mathrm{Pic}\, X \oplus \mathbb{Z})$ always

$L \in \mathrm{Pic}(X)$ acts as $\cdot \otimes L$

$n \in \mathbb{Z}$ acts as $[n]$.

Let X, Y be smooth projective varieties
$E \in D^b(X \times Y)$

$$\begin{array}{ccc} X \times Y & \xrightarrow{q} & Y \\ p \downarrow & & \\ X & & \end{array}$$ - the standard projections

Then we get a functor

$$\Phi_E : D^b(X) \to D^b(Y)$$
$$F \mapsto Rq_*(E \overset{L}{\otimes} Lp^*F)$$

We will call such functors representable.

<u>Theorem (Orlov)</u> Let X, Y - smooth projective and let

$$\Phi : D^b(X) \to D^b(Y)$$

be an exact equivalence. Then $\exists$ a unique up to isomorphism $\mathcal{E} \in D^b(X \times Y)$ s.t.

$$\Phi \cong \Phi_{\mathcal{E}}$$

<u>Corollary</u> If $D^b(X) \cong D^b(Y)$ for X, Y - smooth projective, then $\exists$ isomorphism of $\mathbb{Q}$ vector spaces

$$H^\bullet(X, \mathbb{Q}) \xrightarrow[\sim]{\varphi} H^\bullet(Y, \mathbb{Q})$$

which preserves the Hodge verticals i.e.

$$\varphi\Big(\bigoplus_{p-q=k} H^{p,q}(X)\Big) = \bigoplus_{p-q=k} H^{p,q}(Y)$$

<u>Examples:</u>

(1) Abelian Varieties. Let A be an abelian variety, $\hat{A}$ - the dual abelian variety. The assignment

$$A \mapsto \hat{A}$$

is an involution on the category of abelian varieties

Consider a map

$$f : A \times \hat{A} \to B \times \hat{B}$$

write $f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $\quad \alpha : A \to B$

$\beta : \hat{A} \to B$

etc.

Consider $\tilde{f} : B \times \hat{B} \to A \times \hat{A}$ $\quad \tilde{f} = \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{pmatrix}$

Def: • We say that $f : A \times \hat{A} \to B \times \hat{B}$ is an isometry if f is an isomorphism and if $f^{-1} = \tilde{f}$.

• Let $U(A \times \hat{A}) \subset \mathrm{Aut}(A \times \hat{A})$ be the subgroup of isometries. We will call it Mukai-Polishchuk group.

Thm (Mukai, Polishchuk, Orlov) Let A, B be two abelian varieties over k - an algebraically closed field of char 0, then $D^b(A) \cong D^b(B)$ iff $\exists$ an isometry $f : A \times \hat{A} \to B \times \hat{B}$

Corollary An abelian variety A has only finitely many Fourier-Mukai partners.

Thm (Mukai, Polishchuk, Orlov) There is an exact sequence of groups

$$0 \to \mathbb{Z} \oplus (A \times \hat{A})(k) \to \mathrm{Aut}\, D^b(A) \to U(A \times \hat{A}) \to 0.$$

(2) K3 surfaces over $\mathbb{C}$

Let S be a K3 surface / $\mathbb{C}$
Consider the Mukai lattice of S:
the group

$$\tilde{H}(S, \mathbb{Z}) = \underset{r}{H^0(S, \mathbb{Z})} \oplus \underset{\ell}{H^2(S, \mathbb{Z})} \oplus \underset{\alpha}{H^4(S, \mathbb{Z})}$$

with the Mukai pairing

$$\langle (r_1, \ell_1, \alpha_1), (r_2, \ell_2, \alpha_2) \rangle = r_1 \alpha_2 + r_2 \alpha_1 - \ell_1 \ell_2 .$$

$\tilde{H}(S, \mathbb{C}) \supset H^{2,0}(S)$ is a Hodge structure

Thm (Mukai; Orlov) Two K3 surfaces S_1, S_2 have equivalent derived categories $\iff$ a Hodge isometry of $\tilde{H}(S_1, \mathbb{Z})$ and $\tilde{H}(S_2, \mathbb{Z})$.

Corollary (Bridgeland - Maciocia) $\forall\, S$ - K3 $\Rightarrow \exists$ finitely many FM partners.

<u>Proposition</u> (Hosono-Lian-Oguiso-Yau)
The image
$$\mathrm{Aut}(D^b(S)) \to O_{Hodge}(\tilde{H}(S,\mathbb{Z}))$$
has index at most 2.

(3) The case of minimal elliptic surfaces

(Bridgeland - Maciocia)

(4) Flops

<u>Example</u>: Let $Y = \mathbb{P}^n \subset X$ s.t. $N_{\mathbb{P}^n/X} = \mathcal{O}(-1)^{\oplus n+1}$
Then

$$\begin{tikzcd} & \mathrm{Bl}_Y(X) \arrow[dl, bend right] \arrow[d, "\pi"'] \arrow[r, phantom, "\supset"] \arrow[dr, "\pi^+"] & \mathbb{P}^n \times \mathbb{P}^n \arrow[d] \\ \mathbb{P}^n \subset X & & X^+ \supset \mathbb{P}^n \end{tikzcd}$$

<u>Thm</u> (Bondal, Orlov) $\Phi := R\pi_{+*} L\pi^*$ is an equivalence of derived categories.

<u>Def</u>: Let $f: X \dashrightarrow X^+$ be a birational isomorphism of smooth varieties. Then f is called a generic flop if $\exists$

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi^+"] & \\ X \arrow[rr, dashed, "f"'] & & X^+ \end{tikzcd}$$
s.t.

$$\pi^* \omega_X \cong \pi_+^* \omega_{X_+}.$$

Conjecture: If X and X^+ are related by a generic flop, then

$$D^b(X) \cong D^b(X^+).$$

Note: (1) The conjecture is known in dim $= 3$ due to work of Bridgeland.

(2) the pull-push functor is not necessarily giving the equivalence (Namikawa for Mukai flops).