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Vertex algebras and mirror symmetry

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Let X be a CY, $\omega + iB \in H^{1,1}(X)$

$$\downarrow$$

$N=2$ SCFT.

U vec sp Γ max rank lattice

$I \in \mathrm{End}\, U$. $I^2 = -1$.

$X = T = U/\Gamma$

Theorem 1 Let $(U, I, \Gamma, \omega, B)$, $(U', I', \Gamma', \omega', B')$ be algebraic tori with complexified Kähler classes s.t. the corresp. $N=2$ SCFT are isomorphic. Then

$D^b(T, I)$ and $D^b(T', I')$ are equivalent.

Theorem 2 Let (U, Γ, I) and (U', Γ', I') be algebraic complex tori s.t. $D^b(T, I)$ and $D^b(T', I')$ are equivalent

Then for any $\omega + iB \in H^{1,1}(T)$

$\exists\, \omega' + iB' \in H^{1,1}(T')$

s.t. corresp. $N=2$ SCFT are isomorphic.

Theorem 3 Suppose $(U, I, \Gamma, \omega, B)$, $(U', I', \Gamma', \omega', B')$ are as above, except that $B \in H^2(T)$ and $B' \in H^2(T')$, and $N=2$ SCFTs are isomorphic.

Then $D^b_{\text{twisted}}(T, I, B^{(0,2)})$ and $D^b_{\text{twisted}}(T', I', B'^{(0,2)})$ are equivalent

Analogous theorem to theorem 2 holds.

SCFT V, $Y: V \longrightarrow QF_1(V)$

$QF_1(V)$ elements:

$$\sum_{h \in J} \sum_{n, \bar{n}} \underbrace{C_{(h+n, h+\bar{n})}}_{\in \operatorname{End} V} z^{-(n+h)} \bar{z}^{-(\bar{n}+h)} \qquad J \subset [0, 1)$$

such that

1) $\forall v \in V$

$C_{(h+n, h+\bar{n})} v \neq 0$ for a finite $J_v \subset J$.

2) $\forall v \in V$, $\exists N \in \mathbb{Z}$ s.t.

$C_{(h+n, h+\bar{n})}(v) = 0$ for n or $\bar{n} > N$.

$\underline{QF_2(V)}$:

$$\sum_{(h,g)\in J} \sum_{\substack{n,\bar{n} \\ m,\bar{m} \\ \in \mathbb{Z}}} C_{\text{"index"}} \, z^{-(n+h)} \bar{z}^{-(\bar{n}+h)} w^{-(m+g)} \bar{w}^{-(\bar{m}+g)}$$

$$J \subset [0,1) \times [0,1)$$

s.t. 1) $\forall v \in V$, $C v \neq 0$ for a finite $J_v \subset J$

2) $\forall v \in V$, similar constraint as before.

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data: V, $Y: V \to QF_1(V)$, $|vac\rangle \in V$, $T, \bar{T} \in \text{End}$ (commute)

1) $T|vac\rangle = \bar{T}|vac\rangle = 0$

2) $[T, Y(a,z,\bar{z})] = \partial Y(a,z,\bar{z}) \quad \forall a \in V$

$[\bar{T}, Y(a,z,\bar{z})] = \bar{\partial} Y(a,z,\bar{z})$

3) $Y(a,z,z)|vac\rangle = e^{zT + \bar{z}\bar{T}} a$

4) $\forall a,b \in V$

$$Y(a,z,\bar{z}) Y(b,w,\bar{w}) = \sum_{j=1}^{M} i_{z,w} \frac{1}{(z-w)^{h_j+N}} \, i_{\bar{z},\bar{w}} \frac{1}{(\bar{z}-\bar{w})^{h_j+N}} C_j(z,\bar{z},w,\bar{w}$$

$$(-1)^{p_a p_b} Y(b,w,\bar{w}) Y(a,z,\bar{z}) = \sum_{j=1}^{M} i_{w,z} \frac{1}{(z-w)^{h_j+N}} \, i_{\bar{w},\bar{z}} \frac{1}{(\bar{z}-\bar{w})^{h_j+N}} C_j(z,\bar{z},w,\bar{w})$$

$\boxed{C_j \in QF_2}$.

$$V = H_b \otimes H_f \otimes \mathbb{C}[\Gamma \oplus \Gamma^{*}] , \qquad X = T = U/\Gamma$$

H_b: bosonic Fock space

space of polynomials in a_n^i, $\bar{a}_n^i$, $n = 1, 2, \cdots$, $i = 1, \cdots, \dim U$

H_f: fermionic

N=1 SCFTs associated to flat tori

Let (T, G, B) be a real torus equipped with a flat metric G and a B-field. There is a well-known criterion for two such tori to give equivalent $N = 1$ SCFTs. Let $T = \mathbb{R}^n/\Gamma, T^* = \mathbb{R}^n/\Gamma^*$, and let us define a metric on $T \times T^*$ as follows:

$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$

Let q be a natural even symmetric bilinear form on $\Gamma \oplus \Gamma^*$ given by

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Two tori T and T' give the same $N = 1$ SCFT (i.e. are T-dual) iff there is an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus {\Gamma'}^*$ which intertwines q and q', $\mathcal{G}$ and $\mathcal{G}'$.

N=2 SCFTs associated to complex tori

Given (T, G, B) and a complex structure I on T compatible with G one can define an $N=2$ SCFT. The correponding R-currents are simply

$$J(z) = \omega_{ij}\psi^i(z)\psi^j(z), \bar{J}(\bar{z}) = \omega_{ij}\bar{\psi}^i(\bar{z})\bar{\psi}^j(\bar{z}),$$

where $\omega = GI$ is the Kähler form.

We have found a criterion for (T, I, ω, B) and (T', I', ω', B') to be equivalent or mirror-symmetric. Let us define a pair of commuting complex structures on $T \times T^*$:

$$\begin{aligned}
\mathcal{I}(I, B) &= \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix}, \\
\mathcal{J}(\omega, B) &= \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}.
\end{aligned}$$

Precisely when B is of type $(1,1)$, $\mathcal{I}$ reduces to the obvious product complex structure on $T \times T^*$.

The criteria:

- (T, I, ω, B) and (T', I', ω', B') are equivalent iff there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q', $\mathcal{I}$ and $\mathcal{I}'$, $\mathcal{J}$ and $\mathcal{J}'$.

- (T, I, ω, B) and (T', I', ω', B') are mirror iff there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q', $\mathcal{I}$ and $\mathcal{J}'$, $\mathcal{J}$ and $\mathcal{I}'$.

On the other hand, we have a theorem of Polischuk:

- If both T and T' are algebraic, both B and B' are torsion, and there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma \oplus \Gamma'^*$ which intertwines q and q', $\mathcal{I}$ and $\mathcal{I}'$, then $D^b(\mathfrak{A}_B)$ is equivalent to $D^b(\mathfrak{A}'_{B'})$.

Hence for algebraic tori with torsion B-field isomorphism of $N = 2$ SCFTs implies equivalence of the derived categories of Azumaya algebras. This proves the first part of the generalized Kontsevich's conjecture for algebraic tori.

Remarks:

• I believe that Polischuk's theorem holds also for non-algebraic tori and non-torsion B-fields.

• Our result means that in general B-branes are related to coherent sheaves on a noncommutative "deformation" of X. $Coh(X)$ is relevant iff B is of type $(1,1)$.

• There is a theorem by Orlov which describes when two K3 surfaces have equivalent derived categories. Using his result and some results of Aspinwall and Morrison on the K3 SCFT, one can check that the first part of Kontsevich's conjecture holds for algebraic K3s and B-field of type $(1,1)$.

In the mathematical literature there are some incomplete results on $D^b(\mathfrak{A}_B)$ for K3. The criterion of equivalence of $N = 2$ SCFTs for K3s when B not of type $(1, 1)$ is not completely clear to me either.

We also obtain a converse result:

• If two Azumaya algebras over algebraic tori have equivalent derived categories, then for any flat Kähler metric on the first torus there exists a flat Kähler metric on the second torus such that the corresponding $N = 2$ SCFTs are isomorphic.

Even though we do not yet know the complete definition of the Fukaya category, our results imply:

• Two symplectic tori equipped with a B-field have equivalent Fukaya categories iff there is an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q', $\mathcal{J}$ and $\mathcal{J}'$.