Frobenius manifolds and integrable systems

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Flows of KdV hierarchy $(u_x = u', \dots)$

$$u_{t_0} = u'$$

$$\boxed{u_{t_1} = u u' + \frac{\varepsilon^2}{12} u'''}$$

$$u_{t_2} = \frac{1}{2} u^2 u' + \frac{\varepsilon^2}{12}\left(2u'u'' + u u'''\right) + \frac{\varepsilon^4}{240} u^{V}$$

$$u_{t_3} = \frac{1}{6} u^3 u' + \frac{\varepsilon^2}{24}\left(u'^3 + 4 u u' u'' + u^2 u'''\right) + \frac{\varepsilon^4}{240}\left(5 u'' u''' + 3 u' u^{IV} + u u^{V}\right) + \frac{\varepsilon^6}{6720} u^{VII}$$

$\dots$

commute pairwise

$$\left(u_{t_i}\right)_{t_j} = \left(u_{t_j}\right)_{t_i}$$

Witten-Kontsevich τ-function

of the particular solution

$$u = u(x+t_0, t_1, t_2, t_3, \dots)$$
$$= \partial_x^2 \log \tau (x+t_0, t_1, t_2, t_3, \dots)$$

to the KdV hierarchy w/ initial data

$$u\Big|_{t_0=t_1=\dots=0} = x$$

is the generating function of the Mumford-Morita-Miller classes of $\overline{\mathcal{M}}_{g,n}$

$$\log \tau = \sum_{g\geq 0} \varepsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum_n \frac{1}{n!} t_{p_1} \dots t_{p_n} \int_{\overline{\mathcal{M}}_{g,n}} c_1^{p_1}(\mathcal{L}_1) \wedge \dots \wedge c_1^{p_n}(\mathcal{L}_n)$$

Witten - Kontsevich τ-function

$$\log \tau = \frac{1}{\varepsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \frac{t_0^3 t_1^2}{6} + \frac{t_0^3 t_1^3}{6} + \frac{t_0^3 t_1^4}{6}\right.$$

$$\left. + \frac{t_0^4 t_2}{24} + \frac{t_0^4 t_1 t_2}{8} + \frac{t_0^4 t_1^2 t_2}{4} + \frac{t_0^5 t_2^2}{40} + \frac{t_0^5 t_3}{120} + \dots\right)$$

$$+ \left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0 t_2}{24} + \frac{t_0 t_1 t_2}{12} + \dots\right)$$

$$+ \varepsilon^2\left(\frac{7 t_2^3}{1440} + \frac{7 t_1 t_2^3}{288} + \frac{29 t_2 t_3}{5760} + \frac{29 t_1 t_2 t_3}{1440}\right.$$

$$\left. + \frac{29 t_1^2 t_2 t_3}{576} + \frac{5 t_0 t_2^2 t_3}{144} + \dots\right) + O(\varepsilon^4)$$

$$= \sum \varepsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum_n \frac{1}{n!} t_{p_1} \dots t_{p_n} \int_{\overline{\mathcal{M}}_{g,n}} c_1^{p_1}(\mathcal{L}_1) \wedge \dots \wedge c_1^{p_n}(\mathcal{L}_n)$$

Want to classify integrable systems of 1+1 evolutionary PDEs

$$u^i_t = K^i(u, u_x, u_{xx}, \ldots), \quad i=1,\ldots,n$$
$$u = (u^1(x,t), \ldots, u^n(x,t))$$

w.r.t. generalized Miura transformations

$$u^i \longmapsto \tilde{u}^i = F^i(u, u_x, u_{xx}, \ldots), \quad i=1,\ldots,n$$

Result: the structure of an arbitrary integrable system of 1+1 evolutionary PDEs is controlled by the topology of $\overline{\mathcal{M}}_{g,n}$ (?)

Joint w/ Y. Zhang

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Plan:

- Properties of KdV hierarchy
- Formulation of classification problem
- "Quantum" Riemann-Hilbert problem and explicit construction
- GW invariants

Properties of KdV hierarchy

- Recursive construction
  (bihamiltonian structure)
- τ - structure
- Virasoro symmetries
- quasitriviality

Bihamiltonian structure

$$u_{t_j} = \{u(x), H_j\}_1 = \left(j+\tfrac{1}{2}\right)^{-1} \{u(x), H_{j-1}\}_2$$

$$= \partial_x \frac{\delta H_j}{\delta u(x)}$$

Two compatible Poisson brackets

$$\{u(x), u(y)\}_1 = \delta'(x-y)$$

$$\{u(x), u(y)\}_2 = u(x)\,\delta'(x-y) + \frac{1}{2} u'(x)\,\delta(x-y) + \frac{\varepsilon^2}{8}\,\delta'''(x-y)$$

$$a_1 \{\ ,\ \}_1 + a_2 \{\ ,\ \}_2$$

is a Poisson bracket $\forall\ a_1, a_2$

Hamiltonians $H_j = \int h_j\,(u, u_x, \dots, u^{(2j+2)})\,dx$

$h_{-1} = u$ (Casimir of $\{\ ,\ \}_1$)

$$h_0 = \frac{u^2}{2} + \varepsilon^2 \frac{u''}{12}, \qquad h_1 = \frac{u^3}{6} + \frac{\varepsilon^2}{24}\left(u'^2 + 2uu''\right) + \varepsilon^4 \frac{u^{IV}}{240}$$

$$h_2 = \frac{u^4}{24} + \frac{\varepsilon^2}{24}\left(uu'^2 + u^2u''\right) + \frac{\varepsilon^4}{480}\left(3u''^2 + 4u'u''' + 2uu^{IV}\right) + \frac{\varepsilon^6}{6720}\,u^{VI}$$

τ- function: for any solution $u(x,\vec{t})$ to the KdV hierarchy $\exists$ function $\tau(x,\vec{t})$ s.t.

$$h_k(u(x,\vec{t}), u_x(x,\vec{t}),\dots) = \varepsilon^2 \frac{\partial^2 \log \tau(x,\vec{t})}{\partial x\, \partial t^{k+1}}$$

$k=-1,0,1,2,\dots$, i.e., $\dfrac{\partial h_k}{\partial t_{l+1}} = \dfrac{\partial h_l}{\partial t_{k+1}}$

Generating function

$$\langle x|\, e^{zL}\, |x\rangle \sim \frac{1}{\sqrt{-4\pi z}} \sum_{k=0}^{\infty} z^k h_{k-2}(u, u_x, \dots)$$

$$L = \frac{\varepsilon^2}{2}\partial_x^2 + u$$

$$h_{-2} = 1$$

Virasoro symmetries of KdV hierarchy

$$u \longmapsto u + \delta u$$

solutions $\longmapsto$ solutions

$$\delta u = \varepsilon^2 \partial_x^2 \delta \tau$$

$$\delta \tau = \sum_{m \geq -1} c_m L_m \tau$$

$c_{-1}, c_0, c_1, \ldots$ = const

$$L_{-1} = \sum_{p \geq 1} t_p \partial_{p-1} + \frac{1}{2\varepsilon^2} t_0^2$$

$$L_0 = \sum_{p \geq 0} \left(p + \tfrac{1}{2}\right) t_p \partial_p + \frac{1}{16}$$

$$L_1 = \sum_{p \geq 0} \left(p + \tfrac{1}{2}\right)\left(p + \tfrac{3}{2}\right) t_p \partial_{p+1} + \frac{\varepsilon^2}{8} \partial_0^2$$

$$L_2 = \sum_{p \geq 0} \left(p + \tfrac{1}{2}\right)\left(p + \tfrac{3}{2}\right)\left(p + \tfrac{5}{2}\right) t_p \partial_{p+2} + \frac{3\varepsilon^2}{8} \partial_0 \partial_1$$

$$[L_i, L_j] = (i-j) L_{i+j} + \frac{1}{12} i (i^2 - 1) \delta_{i+j,0}$$

Example: Miura transformation

$$u = v^2 + v_x$$

maps solutions of mKdV

$$v_t = 6v^2 v_x - v_{xxx}$$

to solutions of KdV

$$u_t = 6 u u_x - u_{xxx}$$

Problem: not invertible!

Example: Miura transformation

$$u = v^2 + \varepsilon v_x$$

maps solutions of mKdV

$$v_t = 6 v^2 v_x - \varepsilon^2 v_{xxx}$$

to solutions of KdV

$$u_t = 6 u u_x - \varepsilon^2 u_{xxx}$$

Problem: ~~not invertible~~! $(x \to \varepsilon x,\ t \to \varepsilon t)$

$$v = \sqrt{u} - \frac{\varepsilon}{4} \frac{u'}{u} + \mathcal{O}(\varepsilon^2)$$

Remark

$$\psi = \exp \frac{1}{\varepsilon} \int^x v \, dx$$

is the WKB solution to

$$\varepsilon^2 \psi_{xx} + u \psi = 0$$

$$\psi = u^{-1/4} \, e^{\frac{1}{\varepsilon} \int^x \sqrt{u}\, dx} \left(1 + \mathcal{O}(\varepsilon)\right)$$

Quasitriviality of KdV

Isomorphism

$$v \longmapsto u = v + \frac{\varepsilon^2}{24}\left(\log v'\right)''$$

$$+ \varepsilon^4 \left[\frac{v^{IV}}{1152\, v'^2} - \frac{7\, v'' v'''}{1920\, v'^3} + \frac{v''^3}{360\, v'^4} \right]'' + O(\varepsilon^6)$$

between

dispersionless KdV $\rightsquigarrow$ KdV

$$v_{t_0} = v' \qquad\qquad u_{t_0} = u'$$

$$v_{t_1} = v v' \qquad\qquad u_{t_1} = u u' + \frac{\varepsilon^2}{12} u'''$$

$$\cdots \qquad\qquad \cdots$$

$$v_{t_p} = \frac{v^p}{p!} v' \qquad\qquad u_{t_p} = \frac{u^p}{p!} u' + O(\varepsilon^2)$$

General setting

n-dimensional manifold

$$M^n \ni (u^1, \dots, u^n)$$

Loop space

$$\mathcal{L}_\varepsilon(M^n) = \{ (u^1(x), \dots, u^n(x)) \mid x \in S^1 \} \otimes \mathbb{C}[[\varepsilon]]$$

Jet coordinates $u^i, u^i_x, u^i_{xx}, \dots$

"Changes of coordinates" (Miura group)

$$\mathcal{M} = \{ u^i \longmapsto \tilde{u}^i = \sum_{k \geq 0} \varepsilon^k F^i_{[k]}(u; \underbrace{u_x, \dots, u^{(k)}}_{\substack{\text{polynomial} \\ \text{of deg} = k}}) \}$$

$$\det\left(\partial F^i_{[0]}(u) \big/ \partial u^j\right) \neq 0$$

Dynamical systems on $\mathcal{L}_\varepsilon(M^n)$

$$u^i_t = A^i_j(u)\,u^j_x + \sum_{k\geq 1} \varepsilon^k K^i_{[k]}(u;\underbrace{u_x,\dots,u^{(k+1)}}_{\deg = k+1})$$

system of 1+1 evolutionary PDEs

Poisson brackets (0,n) brackets

$$\{u^i(x), u^j(y)\} = \sum_{k\geq 0} \varepsilon^k \{u^i(x), u^j(y)\}^{[k]}$$

$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{ij}_{ks}(u;\underbrace{u_x,\dots,u^{(s)}}_{\deg = s})\,\delta^{(k-s+1)}(x-y)$$

$$\det\left(A^{ij}_{00}(u)\right) \neq 0, \quad u \in M^n$$

Hamiltonians $H = \int h(u;u_x,\dots;\varepsilon)\,dx$

$$h = \sum_{k\geq 0} \varepsilon^k h^{[k]}(u;\underbrace{u_x,\dots,u^{(k)}}_{\deg = k})$$

Main problem

Classify bihamiltonian structures on $\mathcal{L}_\varepsilon(M^n)$

= compatible pairs $a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$ of $(0,n)$ brackets

Additional assumption: semisimplicity

distinct roots of

$$\det\left(\overset{2}{A}{}^{ij}_{00}(u) - \lambda\, \overset{1}{A}{}^{ij}_{00}(u)\right) = 0$$

for generic $u \in M^n$

Axioms (cf. KdV)

- bihamiltonian
- existence of τ-function
- quasitriviality
- Virasoro symmetries act linearly onto τ-function

Results:

① in the leading order the PDE must have the form

$$v_t = v \cdot v_x \qquad (*)$$

$v = v^1 e_1 + \ldots + v^n e_n \in$ Frobenius manifold

$$e_\alpha \cdot e_\beta \Big|_v = c_{\alpha\beta}^{\gamma}(v)\, e_\gamma$$

associative algebra w/ unity, $c_{1\beta}^{\gamma}(v) = \delta_\beta^\gamma$

$$c_{\alpha\beta}^{\gamma}(v) = \eta^{\gamma\varepsilon} \frac{\partial^3 F(v)}{\partial v^\varepsilon \partial v^\kappa \partial v^\beta}$$

for a quasihomogeneous function $F(v)$, $\quad v = (v^1, \ldots, v^n)$

② If the algebra is semisimple then the system (*) is integrable (completeness of the conservation laws).

③ Under the semisimplicity assumption all the terms of the ε-expansion are uniquely determined by the leading term.

Moduli of integrable PDEs = Moduli of semisimple Frobenius manifolds

④ General monotone solution: $\tau(\vec{t})$ is determined from the linear system of Virasoro constraints

$$L_k\left(\varepsilon^{-1}\left(\vec{t}-\vec{c}(\varepsilon)\right), \varepsilon\frac{\partial}{\partial \vec{t}}\right)\tau = 0, \quad k \geq -1$$

$\vec{t} = (t^{\alpha,p})$, $\alpha = 1, \dots, n$, $p = 0, 1, 2, \dots$

Example Topological solution $\tau_{top}(\vec{t})$:

$c^{\alpha,p} = \delta^\alpha_1 \delta^p_1$

Example Toda

$$u_t = \frac{1}{\varepsilon}\left[v(x) - v(x-\varepsilon)\right]$$

$$v_t = \frac{1}{\varepsilon}\left[e^{u(x+\varepsilon)} - e^{u(x)}\right]$$

(Extended) Toda hierarchy

$$\varepsilon \frac{\partial L}{\partial t^{\alpha,p}} = \left[A_{\alpha,p}, L\right], \quad \alpha = 1,2 \quad p = 0,1,2,\dots$$

$$L = \Lambda + v + e^{u}\Lambda^{-1}$$

$$\Lambda = e^{\varepsilon \partial_x}$$

$\alpha = 2$ Standard Toda

$$A_{2,p} = \frac{1}{(p+1)!}\left[L^{p+1}\right]_+$$

$\alpha = 1$ "second half" of Toda hierarchy

(new)

$$A_{1,0} = \varepsilon \partial_x$$

$$A_{1,1} = \varepsilon \partial_x \circ L - \Lambda + [B_+ - 1] v + [1 - B_-] u \cdot e^u \Lambda^{-1}$$

$$B_\pm = \pm \left[\Lambda^{\pm 1} - 1 \right]^{-1} \varepsilon \partial_x = \sum_{k=0}^{\infty} \frac{B_k}{k!} (\pm \varepsilon \partial_x)^k$$

(G. Carlet)

$$\frac{\partial u}{\partial t^{1,1}} = \frac{1}{\varepsilon} \Big[v (B_- u - 2) - \Lambda^{-1} v (B_- \Lambda u - 2) + B_+ \Lambda v - B_+ \Lambda^{-1} v \Big]$$

$$\frac{\partial v}{\partial t^{1,1}} = v v_x + \frac{1}{\varepsilon} \Big[\Lambda e^u (B_- \Lambda u - 2) - e^u (B_- \Lambda^{-1} u - 2) \Big]$$

GW invariants and their descendents

X smooth projective, $\dim_{\mathbb{C}} X = d$, $H^{odd}(X) = 0$

$$X_{g,m,\beta} = \{ (C_g, x_1, \dots, x_m) \xrightarrow{f} X ,$$

$$f^*[C_g] = \beta \in H_2(X) \}$$

Basis $\phi_1 = 1, \dots, \phi_N$ of $H^*(X; \mathbb{C})$

$$\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_{g,\beta} :=$$

$$= \int_{[X_{g,m,\beta}]^{virt}} ev_{x_1}^*(\phi_{\alpha_1}) \wedge c_1^{p_1}(\mathcal{L}_1) \wedge \dots \wedge ev_{x_m}^*(\phi_{\alpha_m}) \wedge c_1^{p_m}(\mathcal{L}_m)$$

$$\mathcal{F}_g^X(t) = \sum_m \sum_{\beta \in H_2(X)} \frac{1}{m!} t^{\alpha_1, p_1} \cdots t^{\alpha_m, p_m} \langle \tau_{p_1}(\phi_{\alpha_1}) \cdots \tau_{p_m}(\phi_{\alpha_m}) \rangle_{g,\beta}$$

$$\mathcal{F}^X(t;\varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}_g^X(t)$$

Identities in $H^*(\overline{\mathcal{M}}_{g,m})$

$\rightsquigarrow$ differential eqs. for $\mathcal{F}^X(t)$

Denote

$$\langle\!\langle \tau_{p_1}(\phi_{\alpha_1}) \ldots \tau_{p_m}(\phi_{\alpha_m}) \rangle\!\rangle_g := \frac{\partial^m \mathcal{F}_g^X(t)}{\partial t^{\alpha_1, p_1} \cdots \partial t^{\alpha_m, p_m}}$$

Kähler X, $\dim H^{alg}(X) = n$

(k,k) - forms

- Differential ideal $I^{alg}_{GW}(n)$ generated by identities for derivatives of $\mathcal{F}^X$ valid for all X

- Another differential ideal $I_{KdV}(n)$ generated by derivatives of $\log \tau_{top}$

$$\tau_{top} = \hat{G} \prod_{i=1}^{n} \tau_{\text{Kontsevich-Witten}}$$

valid for all values of parameters (μ, R, S, C)

Main Conjecture

$$I_{KdV}(n) = I^{alg}_{GW}(n)$$

Topological recursion relations

$g=0$

$$\langle\langle \tau_p(\phi_\alpha)\tau_q(\phi_\beta)\tau_r(\phi_\gamma)\rangle\rangle_0$$

$$= \langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\rangle\rangle_0 \, \eta^{\nu\mu} \langle\langle \tau_0(\phi_\mu)\tau_q(\phi_\beta)\tau_r(\phi_\gamma)\rangle\rangle_0$$

$g=1$

$$\langle\langle \tau_p(\phi_\alpha)\rangle\rangle_1 = \langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\rangle\rangle_0 \, \eta^{\nu\mu} \langle\langle \tau_0(\phi_\mu)\rangle\rangle_1$$

$$+ \frac{1}{24} \eta^{\nu\mu} \langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\tau_0(\phi_\mu)\rangle\rangle_0$$