LOOP SYMMETRY

IN THE

SIX VERTEX MODEL

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WITH

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WUPPERTAL

BASED ON

1) T. DEGUCHI, K. FABRICIUS, B. McCOY

THE sl_2 LOOP ALGEBRA SYMMETRY OF THE SIX-VERTEX MODEL
J. STAT. PHYS. 102 (2001) 701

2) K. FABRICIUS AND B. McCOY

BETHE'S EQUATION IS INCOMPLETE FOR THE XXZ MODEL AT ROOTS OF UNITY
J. STAT. PHYS. 103 (2001) 647

3) K. FABRICIUS AND B. McCOY

COMPLETING BETHE'S EQUATIONS AT ROOTS OF UNITY
J. STAT PHYS. 104 (2001) 573

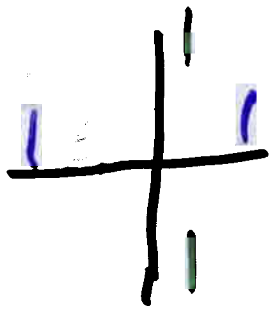
4) K. FABRICIUS AND B. McCOY

EVALUATION PARAMETERS AND BETHE'S ROOTS FOR THE SIX-VERTEX MODEL AT ROOTS OF UNITY
CONDMAT/0108057

OUTLINE

1) WHAT IS A SIX-VERTEX MODEL?
2) WHAT IS A LOOP ALGEBRA?
3) SOME HISTORY
4) SOME EMPERICAL OBSERVATION
5) A CHEVALLEY BASIS
6) SOME REPRESENTATION THEORY
7) BETHE STATES
8) DRINFELD POLYNOMIAL
9) SOME CONSEQUENCES
10) OPEN PROBLEMS

1) SIX-VERTEX MODEL (1967)



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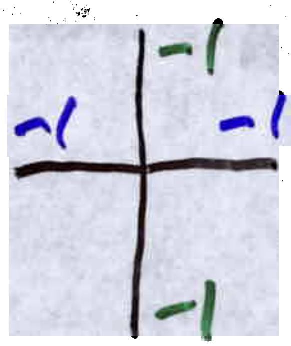
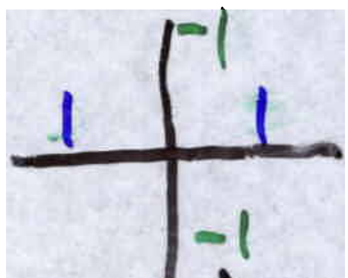


$= \sinh \frac{1}{2}(v + i\gamma)$

$$w(1,1)_{1,1} = w(-1,-1)_{-1,-1}$$



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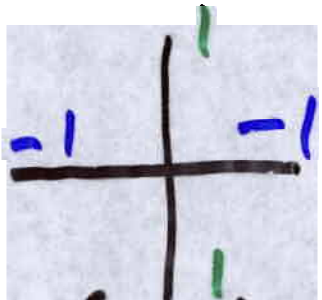
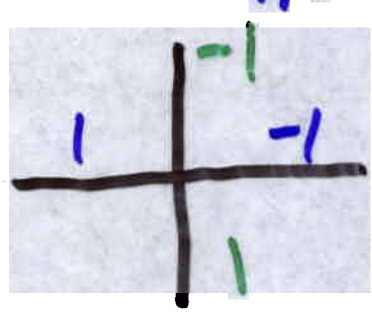


$= \sinh \frac{1}{2}(v - i\gamma)$

$$w(-1,-1)_{1,1} = w(1,1)_{-1,-1}$$



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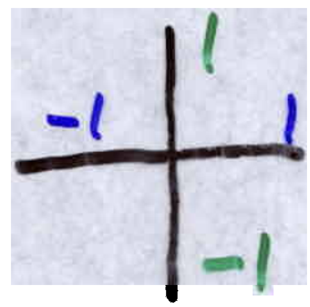
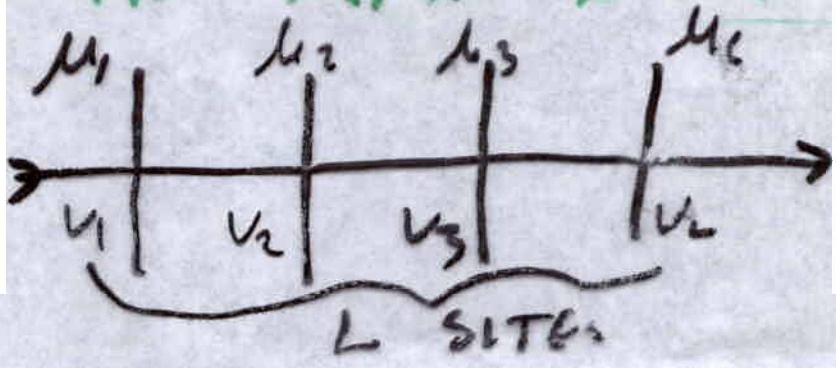


$$w(-1,1)_{1,-1} = w(1,-1)_{-1,1} = \sinh i\gamma$$

$$T(v)_{\mu,\nu} = \mathrm{Tr}\, w(\mu_1,\nu_1)\, w(\mu_2,\nu_2) \cdots w(\mu_L,\nu_L)$$



$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{L}\left\{\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \cos\gamma\,\sigma_j^z\sigma_{j+1}^z\right\}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_j^i = I \otimes \cdots \otimes \underset{\wedge}{\sigma^i} \otimes \cdots \otimes I$$

j POSITION

$[T(v), H_{XXZ}] = 0$ ALL v (MW 1968)

$[T(v), T(v')] = 0$ BAXTER 1971

WE WILL SET

$$q = e^{i\gamma}$$

AT ROOT OF UNITY WE MEAN

$$q_0^{2N} = e^{2i\gamma_0 N} = 1$$

SO $\gamma_0 = \frac{m\pi}{N}$

2) LOOP ALGEBRA (sl_2)

IN THE CHEVALLEY BASIS THIS IS

$$[e_0, f_1] = [f_0, e_1] = 0$$

$$[e_0, h_0] = +e_0 \qquad [e_0, f_0] = h_0$$

$$[f_0, h_0] = -f_0$$

$$[f_1, h_1] = f_1 \qquad [f_1, e_1] = -h_1$$

$$[e_1, h_1] = -e_1$$

WITH $h_0 = -h_1$

AND THE 4 SERRE RELATIONS

$$e_0^3 e_1 - 3 e_0^2 e_1 e_0 + 3 e_0 e_1 e_0^2 - e_1 e_0^3 = 0$$

$$f_0^3 f_1 - 3 f_0^2 f_1 f_0 + 3 f_0 f_1 f_0^2 - 3 f_1 f_0^3 = 0$$

$$f_1^3 f_0 - 3 f_1^2 f_0 f_1 + 3 f_1 f_0 f_1^2 - 3 f_0 f_1^3 = 0$$

$$e_1^3 e_0 - 3 e_1^2 e_0 e_1 + 3 e_1 e_0 e_1^2 - 3 e_0 e_1^3 = 0$$

CONTRAST

1) LOOP sl_2 ALGEBRA

INFINITE DIMENSIONAL ALGEBRA

BUT HAS

FINITE DIMENSIONAL REPRESENTATIONS

WITH

2) $\hat{sl}_2$ (AFFINE sl_2)

IS EXACTLY THE SAME AS

LOOP sl_2 <u>EXCEPT</u> $t_0 \neq -t_1$

IS AN INFINITE DIMENSIONAL ALG

WHICH HAS

<u>NO</u> FINITE DIMENSIONAL REPS

3) SOME HISTORY

1) LOOP ALGEBRA IS FIRST USED BY L. ONSAGER 1944 TO FIND THE EIGENVALUE SPECTRUM OF THE ISING MODEL.

2) THE EIGENVECTORS OF XXZ WERE COMPUTED BY YANG + YANG 1966 USING BETHE'S ANSATZ

3) THE EIGENVECTORS OF 6-VERTEX WERE COMPUTED BY LIEB 1967 (AND SUTHERLAND, YANG, YANG 1967) AND FOUND TO BE IDENTICAL WITH XXZ

4) SOME EMPERICAL OBSERVATIONS

IN 1967 (AND UNTIL QUITE RECENTLY) THERE WAS NO KNOWN CONNECTION BETWEEN ONSAGERS LOOP ALGEBRA SOLUTION AND BETHE'S SOLUTION

IN FACT:

THERE HAS NEVER BEEN ANYONE IN HISTORY WHO HAS DEMONSTRATED IN PRINT THAT THEY HAVE UNDERSTOOD ONSAGERS SOLUTION AT ALL

BUT OVER THE YEARS IT WAS NOTICED BY SEVERAL AUTHORS THAT FOR SPECIAL VALUES OF γ OF THE FORM

$$\gamma_0 = \frac{m\pi}{N} \quad \text{(ROOT OF UNITY)}$$

THE SPECTRUM OF H_{XXZ} (AND 6-VERTEX) HAS SOME DEGENERACIES.

THE ORIGIN OF THESE DEGENERACIES WAS NOT UNDERSTOOD SO THEY WERE CALLED ACCIDENTAL DEGENERACIES

FOR EXAMPLE (SEE PAPER 2)

IF $\ell = 16$ AND $\gamma = \pi/3$

S^2	DEG
8	1
5	4
2	6
−1	4
−4	$\underline{1}$
	$16 = 2^4$

S^2	DEG
7	1
4	4
1	6
−2	4
−5	$\underline{1}$
	$16 = 2^4$

S^2	DEG
6	1
3	4
0	6
−3	4
−6	$\underline{1}$
	$16 = 2^4$

S^2	DEG
5	1
2	2
−1	$\underline{1}$
	$4 = 2^2$

S^2	DEG
5	1
2	3
−1	3
−4	$\underline{1}$
	$8 = 2^3$

S^2	DEG
4	1
1	2
−2	$\underline{1}$
	2^2

S^2	DEG
3	1
0	2
−3	$\underline{1}$
	2^2

S^2	DEG
2	1

S^2	DEG
2	1
−1	$\underline{1}$
	2

S^2	DEG
1	1

S^2	DEG
1	1

THESE DEGENERACIES AND

ALL EXAMPLES ARE SUMMARIZED

BY

$$S^z = S^z_{MAX} - N\ell$$

$$DEG = \binom{2[S^z_{MAX}/N] + (0,1,2)}{\ell}$$

WHERE

$$0 \le \ell \le 2[S^z_{MAX}/N] + (0,1,2)$$

$[X]$ = GREATEST INTEGER IN X

5) A CHEVALLEY BASIS

DEGENERACIES ARE EXPLAINED BY SYMMETRIES. IF $\gamma = 0$ THERE IS A $Sl2$ (NON AFFINE ROTATIONAL SYMMETRY WITH MULTIPLETS LINEAR IN S^2_{max}

NOW AT $\gamma = \gamma_0$ THE MULTIPLETS ARE EXPONENTIAL IN S^2_{max}

SO THERE MUST BE AN AFFINE (INFINITE DIMENSIONAL SYMMETRY.

OUR FIRST RESULT IS THE CHEVALLEY BASIS OF THIS SYMMETRY ALGEBRA

DEFINE

$$S^{\pm(N)} = \sum_{1 \le j_1 < \cdots < j_N \le L} q^{N\sigma^z/2} \otimes \cdots \otimes q^{N\sigma^z/2} \otimes \sigma_{j_1}^{\pm}$$

$$\otimes\, q^{(N-2)\sigma^z/2} \otimes \cdots \otimes q^{(N-2)\sigma^z/2} \otimes \sigma_{j_2}^{\pm}$$

$$\otimes\, \sigma_{j_N}^{\pm} \otimes q^{-N\sigma^z/2} \otimes \cdots \otimes q^{-\sigma^z/2}$$

AND

$T^{\pm(N)} = S^{\pm(N)}$ WITH $q \to q^{-1}$

$$q = e^{i\gamma_0}$$

$$S^z = \frac{1}{2}\sum_{j=1}^{L} \sigma_j^z$$

THEN FOR $S^z \equiv 0 \pmod N$

1) THE OPERATOR $T^{\pm(N)}$, $S^{\pm(N)}$, S^z SATISFY LOOP sl_2 WITH

$e_0 = S^{+(N)}$ $f_0 = S^{-(N)}$ $e_1 = T^{-(N)}$

$f_1 = T^{+(N)}$ $h_0 = -h_1 = -(-q)^N \frac{S^z}{N}$

2) AND

$$[S^{\pm(N)}, H_{XYZ}] = [T^{\pm(N)}, H_{XYZ}] = 0$$

$$S^{\pm(N)} T(v) = q^N T(v) S^{\pm(N)}$$

$$T^{\pm(N)} T(v) = q^N T(v) T^{\pm(N)}$$

WHAT ABOUT THE OTHER SECTOR

$S^z \equiv n \not\equiv 0 \pmod N$?

WE NOW NEED THE ADDITIONAL

$$S^{I} = \sum_{j=1}^{L} q^{\sigma^z/2} \otimes \cdots \otimes q^{\sigma^z/2} \otimes \sigma_j^{I} \otimes q^{-\sigma^z/2} \otimes \cdots \otimes q^{-\sigma^z/2}$$

AND

$T^{\pm} = S^{I}$ WITH $q \to q^{-1}$

NOW EVEN THOUGH $S^{\mp(N)}$ AND $T^{\pm(N)}$ DO **NOT** (ANTI) COMMUTE WITH $T(\lambda)$

FOLLOWING 8 OPERATORS DO

$$(T^+)^n (S^-)^n S^{-(N)} \qquad S^{-(N)} (S^-)^{N-n} T^{+(N-n)}$$

$$S^{+(N)} (S^+)^n (T^-)^n \qquad (T^-)^{N-n} (S^+)^{N-n} S^{+(N)}$$

$$T^{+(N)} (T^+)^n (S^-)^n \qquad (S^-)^{N-n} (T^+)^{N-n} T^{+(N)}$$

$$(S^+)^n (T^-)^n T^{-(N)} \qquad T^{-(N)} (T^-)^{N-n} (S^+)^{N-n}$$

THE OPERATORS LIKE $S^{-1}T^{+1}$
HAVE LARGE NULL SPACES BUT ARE
NOT (FOR $N \geq 3$) PROJECTION OPERATORS
ON THE COMPUTER WE HAVE
CONSTRUCTED PROJECTION OPERATORS
FOR $N=3$ AND VERIFIED THAT
IN THE NON ZERO SUBSPACE
THE Sl_2 LOOP ALGEBRA
IS STILL SATISFIED.

HOW TO PROVE ?

IN A SENSE OUR (ONLY) PROOF IS NOT SATISFACTORY

WE START NOT FROM THE LOOP sl_2 AT $q_0 = e^{i\gamma_0}$

BUT FROM THE QUANTUM GROUP $U_q(\widehat{sl_2})$ AND TAKE THE LIMIT $q \to e^{i\gamma_0}$.

THE LOOP sl_2 THEN EMERGES BY USE OF THE HIGHER ORDER SERRE RELATIONS OF LUSZTIG.

WHERE

$$S^{\pm(n)} = \frac{1}{[N]_q!} S^{\pm N}$$

$$T^{\pm(n)} = \frac{1}{[N]_q!} T^{\pm N}$$

$[N] = \frac{q^N - q^{-N}}{q - q^{-1}}$

6) SOME REPRESENTATION THEORY 18

IT IS NOW "CLEAR" HOW TO STUDY THE MULTIPLETS OF H_{XXZ}. WE MUST DECOMPOSE THE 2^L DIMENSIONAL SPACE OF STATES INTO IRREDUCIBLE REPRESENTATIONS OF LOOP Sl_2 AND THERE IS A THEOREM THAT BECAUSE H IS HERMETIAN THAT THESE FINITE DIMENSIONAL REPRESENTATIONS ARE DIRECT SUMS OF DIRECT PRODUCTS OF EVALUATION REPRESENTATIONS

TO EXPLAIN EVALUATION REPRESENTATION I NEED THE MODE BASIS OF THE LOOP ALGEBRA

$$[e(m), f(n)] = h(m+n)$$
$$[e(m), h(n)] = -2\, e(m+n)$$
$$[f(m), h(n)] = 2\, f(m+n)$$

WITH

$$e(0) = T^{-(N)}, \quad e(-1) = S^{-(N)}$$
$$f(0) = T^{+(N)} \quad f(1) = S^{+(N)}$$

A HIGHEST WEIGHT VECTOR IS DEFINED BY

$$S^{+(N)}|\Omega\rangle = T^{+(N)}|\Omega\rangle = 0$$
$$S^z|\Omega\rangle = S^z_{MAX}|\Omega\rangle$$

EVALUATION REPRESENTATION ARE SPECIFIED BY VECTORS

$$|a_j, m_j\rangle$$

SUCH THAT

$$e(n)|a_j, m_j\rangle = a_j^n e_{m_j} |a_j, m_j\rangle$$
$$f(n)|a_j, m_j\rangle = a_j^n f_{m_j} |a_j, m_j\rangle$$
$$h(n)|a_j, m_j\rangle = a_j^n h_{m_j} |a_j, m_j\rangle$$

a_j ARE CALLED EVALUATION PARAMETERS

e_{m_j}, f_{m_j}, h_{m_j} ARE SPIN $m_j/2$ REPRESENTATIONS OF $s\ell_2$.

FURTHERMORE THE EVALUATION PARAMETERS ARE THE ROOTS OF THE DRINFELD POLYNOMIAL

$$P_\Omega(z) = \prod_j (z - a_j)^{m_j}$$

$$= \sum_{r \geq 0} \mu_r (-z)^r$$

WHERE

$$\frac{T^{+(N)r}}{r!} \frac{S^{-(N)r}}{r!} |\Omega\rangle = \mu_r |\Omega\rangle.$$

IF ALL $m_j = 1$ AND d_Ω IS THE DEGREE OF P_Ω THEN FOR $S^z = S^z_{MAX} - \ell N$

$\binom{d_\Omega}{\ell}$ = NUMBER OF DEGENERATE EIGENVALUES

UNFORTUNATLY I DO NOT KNOW HOW TO COMPUT μ_r FROM THE DEFINITION IN TERMS OF CHEVALLEY GENERATORS.

INSTEAD I WILL CONSIDER THE GENERATING FUNCTION

$$E(z) = \sum_{n=0}^{\infty} e(n) z^{-n}$$

FROM THE DEFINITION

E(z) HAS POLES AT

$$z = q_i$$

7) BETHE STATES

WE NOW NEED TO UNDERSTAND THE SOLUTION OF THE PROBLEM OF THE HIGHEST WEIGHT STATES ORIGINALLY IT IS DONE IN 1967 BUT HERE WE USE THE FORMALISM OF TAKTADJAN AND FADDEEV OF 1979.

DEFINE

$$M_{\lambda,\lambda'}(v,\gamma) = \sum_{\substack{\lambda_i=\pm 1 \\ i=2\cdots L}} W_{\mu_1,\mu_1'}(\lambda,\lambda_2)\cdots W_{\mu_L,\mu_L'}(\lambda_L,\lambda')$$

$$\equiv \begin{array}{c|cc} & + & - \\ \hline + & A(v;\gamma) & B(v;\gamma) \\ - & C(v;\gamma) & D(v;\gamma) \end{array}$$

THEN

$$T(v;\gamma) = A(v;\gamma) + D(v;\gamma)$$

THE EIGENVECTORS ARE

$$|\{v_k\}\rangle = \prod_{k=1}^{n} B(v_k;\gamma)|0\rangle$$

WHERE $|0\rangle = |+ + + \cdots +\rangle$

$$S^z|\{v_k\}\rangle = \left(\frac{L}{2} - n\right)|\{v_k\}\rangle$$

AND v_k SATISFY

BETHE'S EQUATION

$$\left(\frac{\sinh\frac{1}{2}(v_j + i\gamma)}{\sinh\frac{1}{2}(v_j - i\gamma)}\right)^L = \prod_{\substack{\ell=1 \\ \ell\neq j}}^{n} \frac{\sinh\frac{1}{2}(v_j - v_\ell + 2i\gamma)}{\sinh\frac{1}{2}(v_j - v_\ell - 2i\gamma)}$$

AND THE EIGENVALUE

$$t(v) = \sinh^L\tfrac{1}{2}(v - i\gamma)\prod_{j=1}^{n} f(v - v_j) + \sinh^L\tfrac{1}{2}(v + i\gamma)\prod_{j=1}^{n} f(v_j - v)$$

WITH

$$f(v) = -\frac{\sinh(v + 2i\gamma)}{\sinh(v/2)}$$

THE QUESTION CAN NOW BE ASKED IF THIS REALLY DOES GIVE ALL SOLUTIONS.

1) γ "GENERIC" LANGLANDS AND ST. AUBAIN 1997 PROVE THAT IT IS COMPLETE

2) $\gamma = \gamma_0 = \frac{m\pi}{N}$

ON THE COMPUTER IF $\gamma \to \gamma_0$ THERE ARE SETS OF N ROOTS SUCH THAT

$$V_k = v_0 - 2ik\gamma_0 \qquad k = 0, \cdots N-1$$

THESE GIVE FACTORS OF $0/0$ IN THE BETHE EQUATION.

FOR THESE COMPLETE N STRINGS V_0 BECOMES <u>ARBITRARY</u>

MOREOVER THE OPERATOR WHICH "CREATES" THESE STRINGS

$$\prod_{h=0}^{N-1} B(v - 2ih\gamma_0) = 0$$

THEREFORE WE NEED A <u>NEW</u> OPERATOR $B^{(N)}(v)$ TO CREATE THIS N STRING WITH AN ARBITRARY CENTER v. IT IS DEFINED BY

$$T(v) \prod_{j=1}^{\ell} B^{(N)}(v_j') \prod_{h=1}^{n} B(v_h) |0\rangle$$

$$= (-1)^{\ell(N-m)} \prod_{j=1}^{\ell} B^{(N)}(v_j') T(v) \prod_{h=1}^{n} B(v_h) |0\rangle$$

WE HAVE COMPUTED THIS OPERATOR BY TAKING THE LIMIT $\gamma \to \gamma_0$ AND EXPANDING TO FIRST ORDER

$$B^{(N)}(v) = \sum_{k=0}^{N-1} \left(\prod_{l=0}^{k-1} B(v-2il\gamma_0) \right)$$

$$\times \left(B_\gamma(v-2ik\gamma_0) + \frac{X(v-2ik\gamma_0)}{Y(v)} B_v(v-2ik\gamma_0) \right)$$

$$\times \left(\prod_{l=k+1}^{N-1} B(v-2il\gamma_0) \right)$$

WE FIND

$$Y(v) = \sum_{l=0}^{N-1} \frac{\sinh^2 \frac{1}{2}(v-(2l+1)i\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v-v_k-2il\gamma_0) \times \sinh \frac{1}{2}(v-v_k-2i(l+1)\gamma_0)}$$

$$X(v) = \sum_{l=0}^{N-1} \frac{l \sinh^2 \frac{1}{2}(v-(2l+1)i\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v-v_k-2il\gamma_0) \times \sinh \frac{1}{2}(v-v_k-2i(l+1)\gamma_0)}$$

8) DRINFELD POLYNOMIAL

WE NOW COMBINE OUR BETHE'S COMPUTATION WITH REPRESENTATION THEORY TO WRITE

$$e^-(z) = \mathcal{N} B^{(n)}(n)$$

WHERE $\mathcal{N}$ IS SOME NORMALIZING FACTOR.

THE CURRENT $B^{(n)}(n)$ HAS POLES AT V_j . WHERE

$$Y(N_j) = 0$$

THUS DEFINING $z = e^{N\sigma}$ AND $d_\pm$ BY

$$Y(n) \sim C_\pm e^{\pm N d_\pm \sigma} \quad \text{as } N \to \pm\infty$$

WE HAVE THE DRINFELD POLYNOMIAL

$$P_r(z) = e^{d_- N v} Y(v)$$

OF DEGREE

$$d = d_+ + d_-$$

9) SOME CONSEQUENCES

1) ON THE COMPUTER ALL ZEROES OF $P_n(z)$ ARE SIMPLE. THIS THEN PROVES THE MULTIPLICITY FORMULA

2) THE DEGREES d_+ + d_- DEPEND ON THE V_k. THERE ARE SEVERAL CASES

A) NO $V_k = \infty$

$$d_+ = d_- = \left[\frac{S^2}{N}\right]$$

SO

$$d = d_+ + d_- = 2\left[\frac{S^2}{N}\right]$$

B) THE PAIR $v = +\infty$ + $-\infty$ OCCURS p TIMES

FOR $n \geq 3$ WE FIND

$$\frac{L}{2} - n + p \equiv 0 \pmod{N}$$

$$d_+ = d_- = \left[\frac{S^2}{N}\right] + 1$$

$$d = 2\left[\frac{S^2}{N}\right] + 2$$

C) A SINGLE ROOT $v_h = \infty$ or $-\infty$ OCCURS S TIMES

IN THIS CASE WE FIND FOR $N=3$

$$d = 2\left[\frac{S^2}{N}\right] + 1$$

3) SUM RULES FOR BETHE ROOTS

IF $Y(v)$ IS A CONSTANT THERE ARE NO EVALUATION PARAMETERS. NEVERTHELESS $Y(v)$ STILL PROVIDES USEFUL INFORMATION. BECAUSE IT IS INDEPENDENT OF v IT MAY BE EVALUATED AS

$$\text{CONST} = \lim_{v \to +\infty} Y(v) = \lim_{v \to -\infty} Y(v).$$

CURIOUSLY ENOUGH THESE 2 LIMITS ARE NOT TRIVIALLY IDENTICAL AND LEAD TO CONSTRAINTS ON v_h

$S^z = 0$

$$\sum_{k=0}^{L/2} v_k = 0, \; m\pi i$$

$S^z = 1$

$$\left\{L - (e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{\frac{L}{2}-1} e^{v_k}\right\} \prod_{k=1}^{\frac{L}{2}-1} e^{v_k}$$

$$= \left\{L - (e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{\frac{L}{2}-1} e^{-v_k}\right\} \prod_{k=1}^{\frac{L}{2}-1} e^{-v_k}$$

$S^z = 2$

$$\left\{\tfrac{1}{2}L(L-1) - L(e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{\frac{L}{2}-2} e^{v_k}\right.$$

$$+ \tfrac{1}{2}(e^{2i\gamma} + e^{-2i\gamma}) \sum_{k=1}^{\frac{L}{2}-2} e^{2v_k}$$

$$\left. + \tfrac{1}{2}(e^{i\gamma} + e^{-i\gamma})^2 \left(\sum_{k=1}^{\frac{L}{2}-2} e^{v_k}\right)^2\right\} \prod_{k=1}^{\frac{L}{2}-2} e^{v_k}$$

$$= \; v_k \leftrightarrow -v_k$$

VALID FOR $\underline{\text{ALL}}$ γ, NOT JUST γ_0

10) OPEN PROBLEMS

1) REPRESENTATION THEORY SEEMS TO SAY THAT $B^{(n)}(N)$ MUST BE IN THE FORM

$$B^{(n)}(N) = \sum_j \frac{R_j}{N - N_j}$$

WHERE

$$Y(N_j) = 0.$$

AND

$$R_j = \sum_{k=0}^{N-1} \left(\prod_{l=0}^{k-1} B(N_j - 2i\, l\gamma_0)\right) \frac{X(N_j - 2i\, k\gamma_0)}{Y'(N_j)} B_N(N_j - 2i\, k\gamma_0) \left(\prod_{l=k+1}^{N-1} B(N_j - 2i\, l\gamma_0)\right)$$

THIS DOES NOT INVOLVE B_X.

PROVE:

$$\left\{B^{(n)}(N) - \sum_j \frac{R_j}{N - N_j}\right\}|\Omega\rangle = 0 \quad \text{AND} \quad R_j^2 = 0$$

2) COMPUTE THE COMPANION OPERATORS $C^{(N)}(N)$, $H^{(N)}(N)$ AND DEMONSTRATE DIRECTLY THAT THEY SATISFY THE CURRENT FORM OF THE ALGEBRA

3) THE XYZ MODEL

$$H_{XYZ} = -\frac{1}{2}\sum_{j=1}^{L}\left\{ J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right\}$$

ALSO HAS A ROOT OF UNITY CONDITION AND DEGENERATE MULTIPLETS. GENERALIZE TO THIS CASE OF ELLIPTIC ALGEBRA.