Some Mathematical Problems in Conformal Field Theory

Berkeley, 15 April 2002

2D Conformal Field Theory : Quantum field theory in two dimensions covariant under conformal transformations (local dilatations)

Relevant in two areas of Physics

- systems with scale invariance : critical point, long distance behaviour
- string theory : (residual reparametrization invariance)

Fascinating for the physicist : unconventional “non-perturbative” techniques in QFT; connections with many other areas of M∩Φ

Fascinating for the mathematician

– many well posed problems

– multiplicity of concepts involved

- infinite dim Lie algebras (and related topics)
- modular properties
- classical and quantum integrable systems
- “quantum” (=deformed) algebras
- combinatorics

etc etc

– multiplicity of possible languages : algebra and representation theory; geometry (group manifold); topology and TFT; operator algebras (and subfactors); category theory. . .

In this talk, focus on the simplest class of CFT, the “Rational” CFT (RCFT) and on three types of algebras which appear there

(1) infinite dimensional Lie algebras (affine Kac-Moody and Virasoro) and their representations (and modular properties): building blocks of CFT

(2) fusion algebras and their “nimreps” (= Non-negative Integer-valued Matrix Representations) $\leftarrow$ various types of boundary conditions

(3) quantum algebras, constructed from the combinatorial data of (2)

Issue Classification programme of RCFT

but Boundary conditions in CFT relevant in study of “quantum impurities” etc, and of D-branes

A crash course of QFT !!

1. Functional integral approach

Σ space-time with (pseudo-)Riemannian metric γ. Basic objects in QFT are

- partition function $Z_{\Sigma,\gamma} = \int$ "$D\phi$" $e^{-S_{\Sigma,\gamma}[\phi]}$, for ex. field $\phi \in \mathrm{Map}(\Sigma, M)$, M "target" manifold, $S_{\Sigma,\gamma}[\phi]$ local functional $\int_\Sigma L(\phi, \partial\phi, \cdots)$ of ϕ; and

- correlation functions

$$\langle \phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle = Z^{-1}\int D\phi\, e^{-S[\phi]}\,\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)$$

2. Quantum mechanical approach

$\Sigma = \Sigma' \times [0,T]$, Hilbert space $H_{\Sigma'}$ of state vectors $|.\rangle$; vacuum state $|0\rangle \in H$; field operators $\hat{\phi}(\mathbf{x},0)$ act in H ; $\hat{\phi}(\mathbf{x},t) = e^{Ht}\,\hat{\phi}(\mathbf{x},0)\,e^{-Ht}$; $Z = \langle 0|e^{-HT}|0\rangle$;

$\langle \phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle = Z^{-1}\langle 0|T\,e^{-HT}\hat{\phi}_{i_1}(x_1)\cdots\hat{\phi}_{i_n}(x_n)|0\rangle$, T : time ordering.

2d-CFT : Σ a compact (orientable, no boundary) 2-d manifold. Demand that the correlation functions be covariant under analytic changes of local coordinates $(z,\bar{z}) \to (z',\bar{z}')$

$$\begin{aligned} \langle \phi_{i_1}(z'_1,\bar{z}'_1)\cdots\phi_{i_n}(z'_n,\bar{z}'_n)\rangle &= \prod_i \left(\frac{dz_i}{dz'_i}\right)^{h_i} \times (\text{same with } z \to \bar{z}, h \to \bar{h}) \\ &\times \langle \phi_{i_1}(z_1,\bar{z}_1)\cdots\phi_{i_n}(z_n,\bar{z}_n)\rangle \end{aligned}$$

ϕ_i “primary field” of conformal weights $(h_i,\bar{h}_i)$

Infinitesimal form $z \to z' = z + \varepsilon z^{n+1}$ represented by Virasoro generators L_n, $\bar{z} \to \bar{z}'$ by $\bar{L}_n$, 2 commuting copies of Virasoro algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [\bar{L}_n,\bar{L}_m] &= \textit{ibid.} \\ [L,\bar{L}] &= [c,L] = [c,\bar{L}] = 0 \end{aligned}$$

More generally, extend Vir to chiral algebra $\mathfrak{A}$ “containing” it

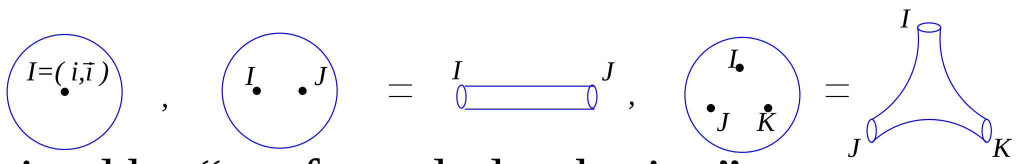
Simplest (and best defined): Extend Vir to affine Lie algebra $\widehat{\mathfrak{g}}$: Vir $\subset U(\widehat{\mathfrak{g}})$ (Sugawara construction : Vir generators quadratic in generators of $\widehat{\mathfrak{g}}$).

Other extensions could include W-algebras, super-conformal extensions, . . .

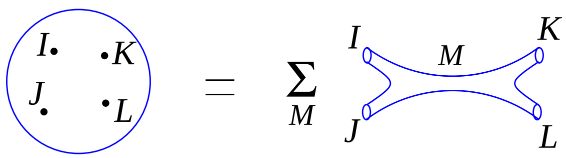
first step study of “chiral data”: representations of chiral algebra $\mathfrak{A}$ etc
second step how to match left and right representations to make a physically sensible theory

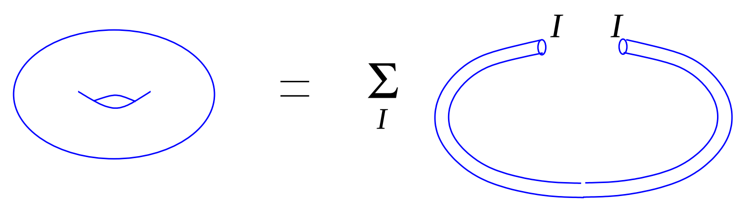
Consistency of CFT

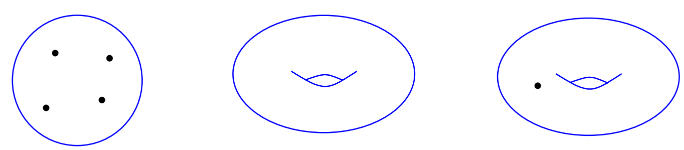
required on Riemann surface of any genus (and with possible boundaries)

Given 

all the others may be obtained by "conformal plumbering"





Consistent and unambiguous if  are.

[Sonoda, Moore & Seiberg] Also similar consistency conditions on surfaces with boundary [Lewellen].

[K. Gawedzki, lectures at Princeton QFT 96-97 programme]

Chiral data of RCFT

Chiral algebra $\mathfrak{A} \supseteq \mathrm{Vir}$, c central charge of Vir

Rationality : finite set of (highest weight) irreps $V_i,\, i \in J$ of $\mathfrak{A}$

$i = 1$ "vacuum repn": lowest eigenvalue of $L_0 = 0$, conjugation $i \to i^*$

Characters $\chi_i(q) = \mathrm{tr}_{V_i}\, q^{L_0 - c/24} = q^{h_i - c/24} \sum_{n=0}^{\infty} \mathrm{mult}(n) q^n$

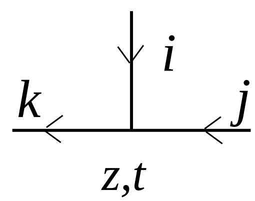
($\to$ unspecialised characters?)

and their modular transformation: $q = \mathrm{e}^{2\pi i \tau}$ $\chi_i(-1/\tau) = \sum_j S_{ij} \chi_j(\tau)$,

S symmetric unitary. T ($\tau \to \tau + 1$) diagonal unitary.

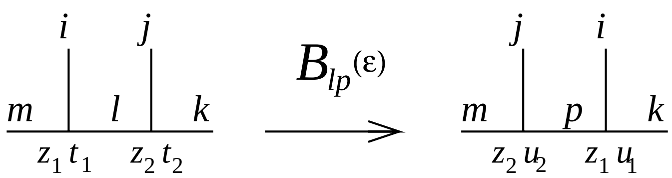
Fusion of representations $V_i \star V_j = \oplus_k {N_{ij}}^k \otimes V_k$

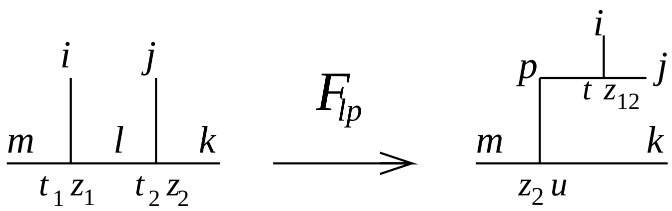
Verlinde formula ${N_{ij}}^k = \sum_{l \in J} \frac{S_{il}}{S_{1l}} S_{jl} S^*_{kl} \in \mathbb{N}$

$=$ dim of space of CVO's $\phi_i(z) \;:\; V_j \mapsto V_k$ 

$t = 1, \cdots, {N_{ij}}^k$

Braiding and Fusing of CVO's





Example: $\mathfrak{A} = \widehat{\mathfrak{g}}_k$, J the integrable weights at level k, S: [Kac-Peterson], F the $6-j$ symbols of $U_q(\mathfrak{g})$, $q = \exp 2i\pi/(k+h^\vee)$

In particular $\widehat{sl}(2)_k$, $J = \{1, 2, \cdots, k+1\}$, $S_{ij} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi i j}{k+2}$

Question Which chiral algebras lead to such properties?

Spectrum of RCFT on plane or half-plane

In the **plane** (or infinite cylinder, $z = \mathrm{e}^{2i\pi w}$), two copies of chiral algebra $\mathfrak{A}$ act on Hilbert space $H = \bigoplus_{\substack{j,\bar{j}\in J \\ \alpha=1,\cdots,Z_{j\bar{j}}}} (V_j \otimes \overline{V}_{\bar{j}})^{(\alpha)}$

Cardy : a system of finite size encodes nicely the spectrum, and its consistency puts constraints on this spectrum.

On a **torus** $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $q = e^{2i\pi\tau}$, the partition function reads

$$Z = \mathrm{tr}_H\, q^{L_0 - c/24} q^{*\bar{L}_0 - c/24} = \sum_{j\bar{j}} Z_{j\bar{j}} \chi_j(q)\, \left(\chi_{\bar{j}}(q)\right)^*$$

is modular invariant, [Cardy 1986] $Z_{j\bar{j}} \in \mathbb{N}$, $Z_{11} = 1$ (unicity of vacuum). $Z_{j\bar{j}} = \delta_{j\bar{j}}$ always a solution. Other solutions?

$\rightarrow$ Classification of Modular Invariants $Z_{j\bar{j}}$?

Theorem [Cappelli-Itzykson-Z., Kato, 1987] *There is a one-to-one correspondence between $\widehat{sl}(2)$ modular invariants of level k and $A-D-E$ Coxeter-Dynkin diagrams of Coxeter number $h=k+2$.*

$k \geq 0$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	(A_{k+1})
$k=4\rho \geq 4$	$\sum_{\lambda \text{ odd } =1}^{2\rho-1} \lvert\chi_\lambda+\chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$(D_{2\rho+2})$
$k=4\rho-2 \geq 6$	$\sum_{\lambda \text{ odd } =1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda \text{ even } =2}^{2\rho-2} (\chi_\lambda\chi^*_{4\rho-\lambda} + \text{c.c.})$	$(D_{2\rho+1})$
$k=10$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	(E_6)
$k=16$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2$ $+[(\chi_3+\chi_{15})\chi_9^* + \text{c.c.}]$	(E_7)
$k=28$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	(E_8)

Theorem [Gannon 1994] List of $\widehat{sl}(3)_k$ modular invariants; $n = k+3$

$(A^{(n)})$ $\quad Z = \sum_{\lambda \in P_{++}^{(n)}} |\chi_\lambda|^2$

$(A^{(n)*})$ $\quad Z = \sum_{\lambda \in P_{++}^{(n)}} \chi_\lambda {\chi_{\lambda^*}}^*$

$(D^{(n)})$ $\quad Z = \begin{cases} \frac{1}{3} \sum_{\lambda \in Q \cap P_{++}^{(n)}} |\chi_\lambda + \chi_{\sigma\lambda} + \chi_{\sigma^2\lambda}|^2 & \text{if 3 divides } n \\ \sum_{\lambda \in Q \cap P_{++}^{(n)}} |\chi_\lambda|^2 + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^{n\tau}\lambda} & \text{if 3 does not divide } n \end{cases}$

$(D^{(n)*})$ $\quad Z = \begin{cases} \frac{1}{3} \sum_{\lambda \in Q \cap P_{++}^{(n)}} \left(\sum_{\ell=0}^{2} \chi_{\sigma^\ell \lambda} \right) \left(\sum_{\ell=0}^{2} \chi_{\sigma^\ell \lambda^*} \right)^* & \text{if 3 divides } n \\ \sum_{\lambda \in Q \cap P_{++}^{(n)}} \chi_\lambda \chi_{\lambda^*} + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^{-n\tau}\lambda^*} & \text{if 3 does not divide } n \end{cases}$

$(E^{(8)})$ $\quad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + |\chi_{(3,2)} + \chi_{(1,6)}|^2 + |\chi_{(2,3)} + \chi_{(6,1)}|^2 + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + |\chi_{(1,3)} + \chi_{(4,3)}|^2 + |\chi_{(3,1)} + \chi_{(3,4)}|^2$

$(E^{(8)*})$ $\quad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + \left(\chi_{(3,2)} + \chi_{(1,6)}\right)\left(\chi_{(2,3)} + \chi_{(6,1)}\right)^* + \text{c.c.} + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + \left(\chi_{(1,3)} + \chi_{(4,3)}\right)\left(\chi_{(3,1)} + \chi_{(3,4)}\right)^* + \text{c.c.}$

$(E^{(12)})$ $\quad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2$

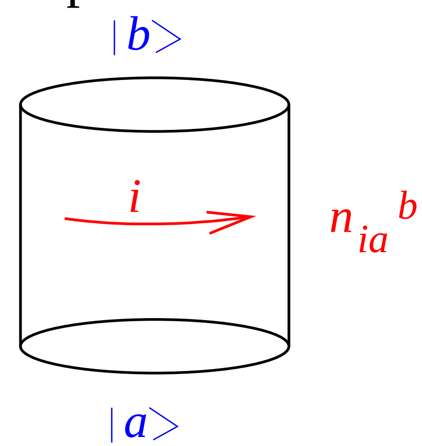
$(E^{(12)}_{MS})$ $\quad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + |\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}|^2$
$\quad + |\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}|^2 + 2|\chi_{(4,4)}|^2 + \left(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\right)\chi^*_{(4,4)} + \text{c.c.}$

$(E^{(12)*}_{MS})$ $\quad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(4,4)}|^2$
$\quad + \left(\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}\right)\left(\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}\right)^* + \text{c.c.} + \left(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\right)\chi^*_{(4,4)} + \text{c.c.}$

$(E^{(24)})$ $\quad Z = |\chi_{(1,1)} + \chi_{(22,1)} + \chi_{(1,22)} + \chi_{(5,5)} + \chi_{(5,14)} + \chi_{(14,5)} + \chi_{(11,11)} + \chi_{(11,2)} + \chi_{(2,11)} + \chi_{(7,7)} + \chi_{(7,10)} + \chi_{(10,7)}|^2$
$\quad + |\chi_{(7,1)} + \chi_{(16,7)} + \chi_{(1,16)} + \chi_{(1,7)} + \chi_{(7,16)} + \chi_{(16,1)} + \chi_{(5,8)} + \chi_{(11,5)} + \chi_{(8,11)} + \chi_{(8,5)} + \chi_{(5,11)} + \chi_{(11,8)}|^2$

In the upper **half-plane**, with boundary conditions a and b on the $\gtrless 0$ real axis, only one copy of Vir or $\mathfrak{A}$ acts and $H_{ab} = \oplus_{i\in J}\, n_{ia}{}^{b} V_i$

Take a finite **annulus** and demand consistency with other picture in terms of boundary states $|a\rangle$, $|b\rangle$ in H satisfying $(L_n - \overline{L}_{-n})|a\rangle = 0$ etc. Find **Cardy equation** :



$$n_{ia}{}^{b} = \sum_{j,\alpha=1,\cdots,Z_{jj}} \frac{S_{ij}}{S_{1j}} \psi_a^{(j,\alpha)} \psi_b^{(j,\alpha)*}$$

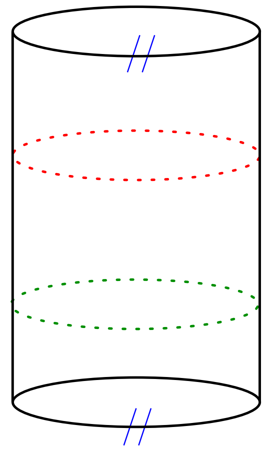
Compare with Verlinde $\quad N_{ij}{}^{k} = \sum_{l\in J} \frac{S_{il}}{S_{1l}} S_{jl} S^*_{kl}$

Assume ψ unitary $\Rightarrow$ $n_i n_j = \sum_k N_{ij}{}^{k} n_k\,, \qquad (n_i)^T = n_{i^*}$

Proposition [Behrend, Pearce, Petkova,Z. 1998] Complete orthonormal solutions of Cardy equation $\leftrightarrow$ nimreps of the fusion algebra subject to $(n_i)^T = n_{i^*}$ and spectrum specified by (diagonal part of) mod. invariant Z_{jj}

$\rightarrow$ Classification of Nimreps n ?

Torus, but with defect lines (or "seams")



Look for (non local) operators X in H that commute with $\mathfrak{A} \otimes \bar{\mathfrak{A}}$

$$[L_n, X] = [\bar{L}_n, X] = 0$$

(Invariant under deformation of contour)

Schur's lemma $\Rightarrow$ complete characterization of these X in terms of projectors between equivalent copies
$(V_j \otimes \bar{V}_{\bar{j}})^{\alpha'} \mapsto (V_j \otimes \bar{V}_{\bar{j}})^{\alpha}, \qquad \alpha, \alpha' = 1, \cdots, Z_{j\bar{j}}.$
Independent X_x labelled by $x = 1, \cdots, \sum Z^2_{j\bar{j}}, \qquad X_1 = I.$

Alternative picture in which Hilbert space

$$H_{x|y} = \oplus_{i,\bar{i}\in J} \widetilde{V}_{i\bar{i}^*;x}{}^{y}\, V_i \otimes \overline{V}_{\bar{i}}\,,$$

(and in particular $\widetilde{V}_{i\bar{i}^*;1}{}^{1} = Z_{i\bar{i}}$)

Consistency condition

$$\widetilde{V}_{i_1 j_1}\widetilde{V}_{i_2 j_2} = \sum_{i_3,j_3} N_{i_1 i_2}{}^{i_3} N_{j_1 j_2}{}^{j_3}\, \widetilde{V}_{i_3 j_3}$$

Proposition [Petkova, Z. 2000] The $\widetilde{V}$ form a nimrep of the double fusion algebra.

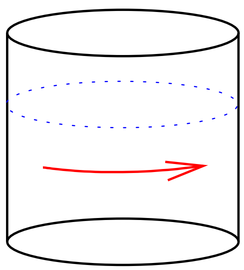
(related to modular invariant by spectrum and by condition $\widetilde{V}_{i\bar{i}^*;1}{}^{1} = Z_{i\bar{i}}$.)

$\rightarrow$ Classification of Nimreps $\widetilde{V}$ of Double Fusion Algebra ?

Set of x : Ocneanu “quantum symmetries”

Cylinder and Defect Line

Similar story $H_{ax|b} = \oplus (n_i\, \tilde{n}_x)_{a^*}{}^{b^*}\, V_i$



$$\tilde{n}_{ax}{}^{b} = \sum_{j,\alpha,\beta} \psi_a^{(j,\alpha)} \frac{\Psi_x^{(j,j;\alpha,\beta)}}{\Psi_1^{(j,j)}} \psi_b^{(j,\beta)*}$$

The $\tilde{n}$ form a nimrep of a new, associative, but in general non-commutative, algebra

$$\tilde{n}_x \tilde{n}_y = \sum_z \widetilde{N}_{xy}{}^z \tilde{n}_z$$

$$X_x X_y = \sum_z \widetilde{N}_{yx}{}^z X_z \qquad \widetilde{N}_{yx}{}^z \in \mathbb{N}$$

Fusion of defect lines

Summary

A set of multiplicity matrices, forming nimreps of fusion algebras

$$
\begin{aligned}
N_i N_j &= \sum_k N_{ij}{}^k N_k \\
n_i n_j &= \sum_k N_{ij}{}^k n_k \\
\widetilde{V}_{i_1 j_1} \widetilde{V}_{i_2 j_2} &= \sum_{i_3, j_3} N_{i_1 i_2}{}^{i_3} N_{j_1 j_2}{}^{j_3} \, \widetilde{V}_{i_3 j_3} \qquad (*) \\
\tilde{n}_x \tilde{n}_y &= \sum_z \widetilde{N}_{xy}{}^z \tilde{n}_z \\
\widetilde{N}_x \widetilde{N}_y &= \sum_z \widetilde{N}_{xy}{}^z \widetilde{N}_z \,.
\end{aligned}
$$

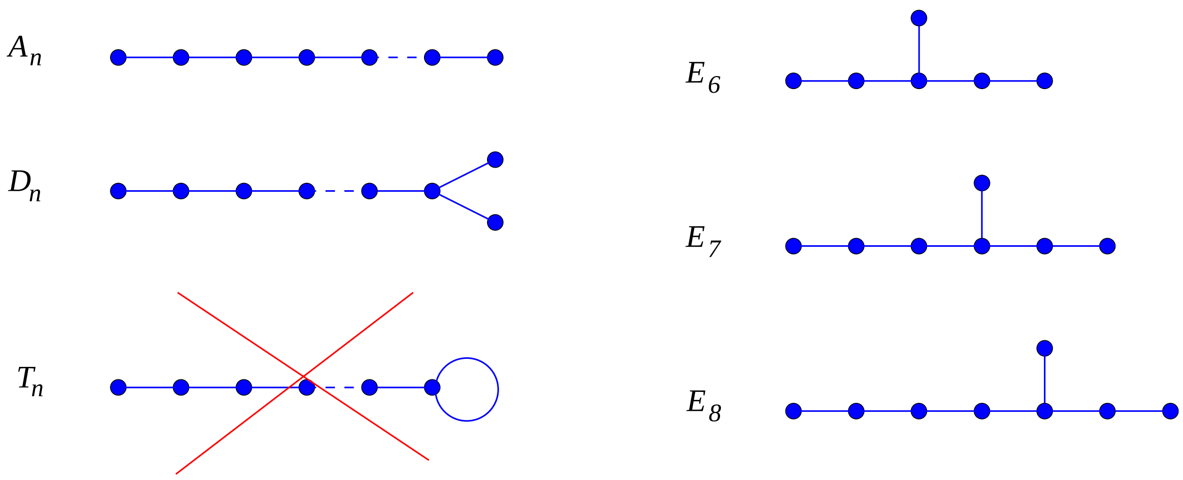
where $(N_i)_j{}^k = N_{ij}{}^k$, $(\widetilde{N}_y)_x{}^z = \widetilde{N}_{yx}{}^z$, plus some constraints: spectrum, n_i, $\tilde{n}_x$ commute etc.

“Trivial” solution when $Z_{ij} = \delta_{ij} : N_i = n_i = \tilde{n}_i = \widetilde{N}_i$, $\widetilde{V}_{ij} = N_i N_j$

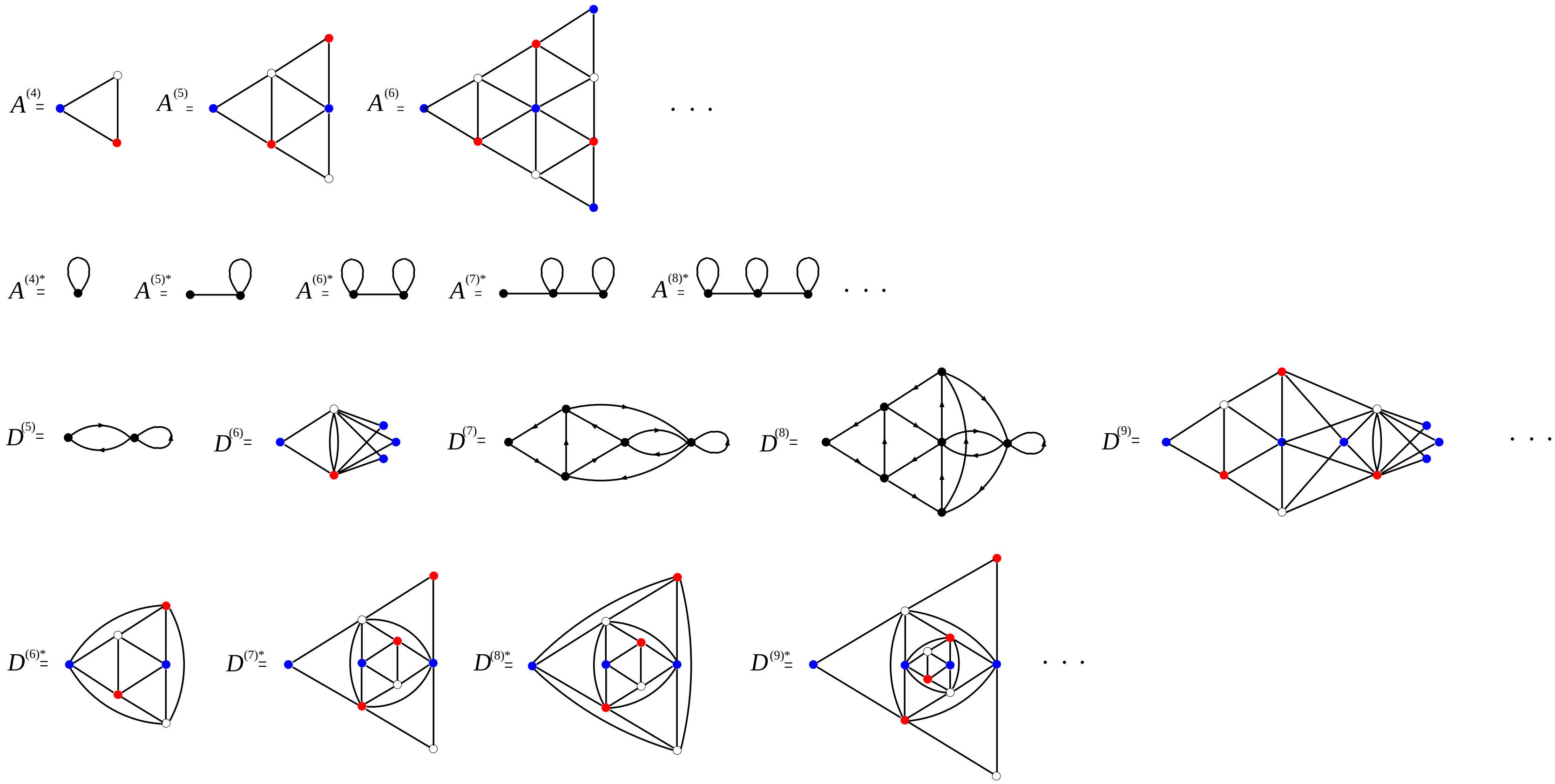
From Nimreps to Graphs

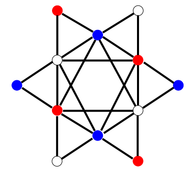
These (integer valued) matrices are conveniently encoded as graphs. Sufficient to draw the graph corresponding to the generator(s) n_f of these algebras. For example, for $\widehat{sl}(2)$, N_2, n_2, $(\widetilde{V}_{21}, \widetilde{V}_{12})$.

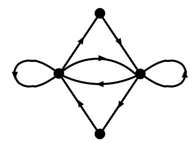
Graphs of n_f ? For $\widehat{sl}(2)$, n_2 must have eigenvalues $2\cos\pi j/(k+2)$, hence a graph of $A-D-E$ or "tadpole" type. Tadpoles ruled out to match ADE classification of modular invariants

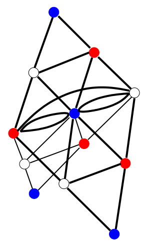


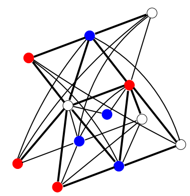
For $\widehat{sl}(3)$ ……Kostov; DiFrancesco, Petkova, Behrend, Pearce, Z; Ocneanu so as to match the list of modular invariants of Gannon

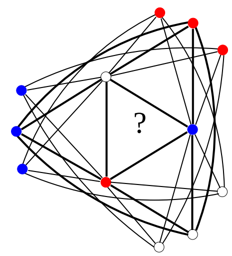


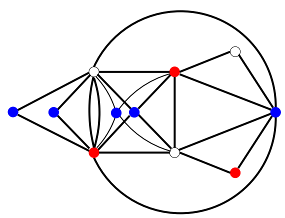
$E^{(8)} =$ 

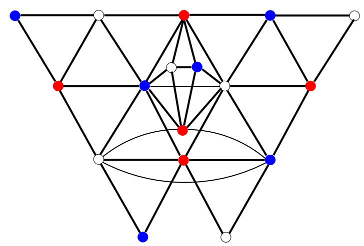
$E^{(8)*} =$ 

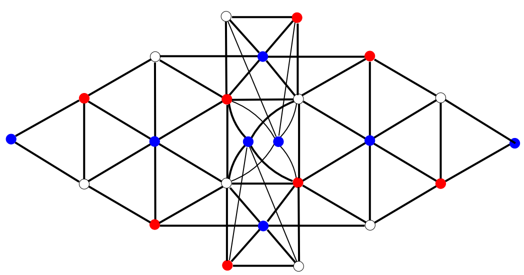
$E_1^{(12)} =$ 

$E_2^{(12)} =$ 

$E_3^{(12)} =$ 

$E_4^{(12)} =$ 

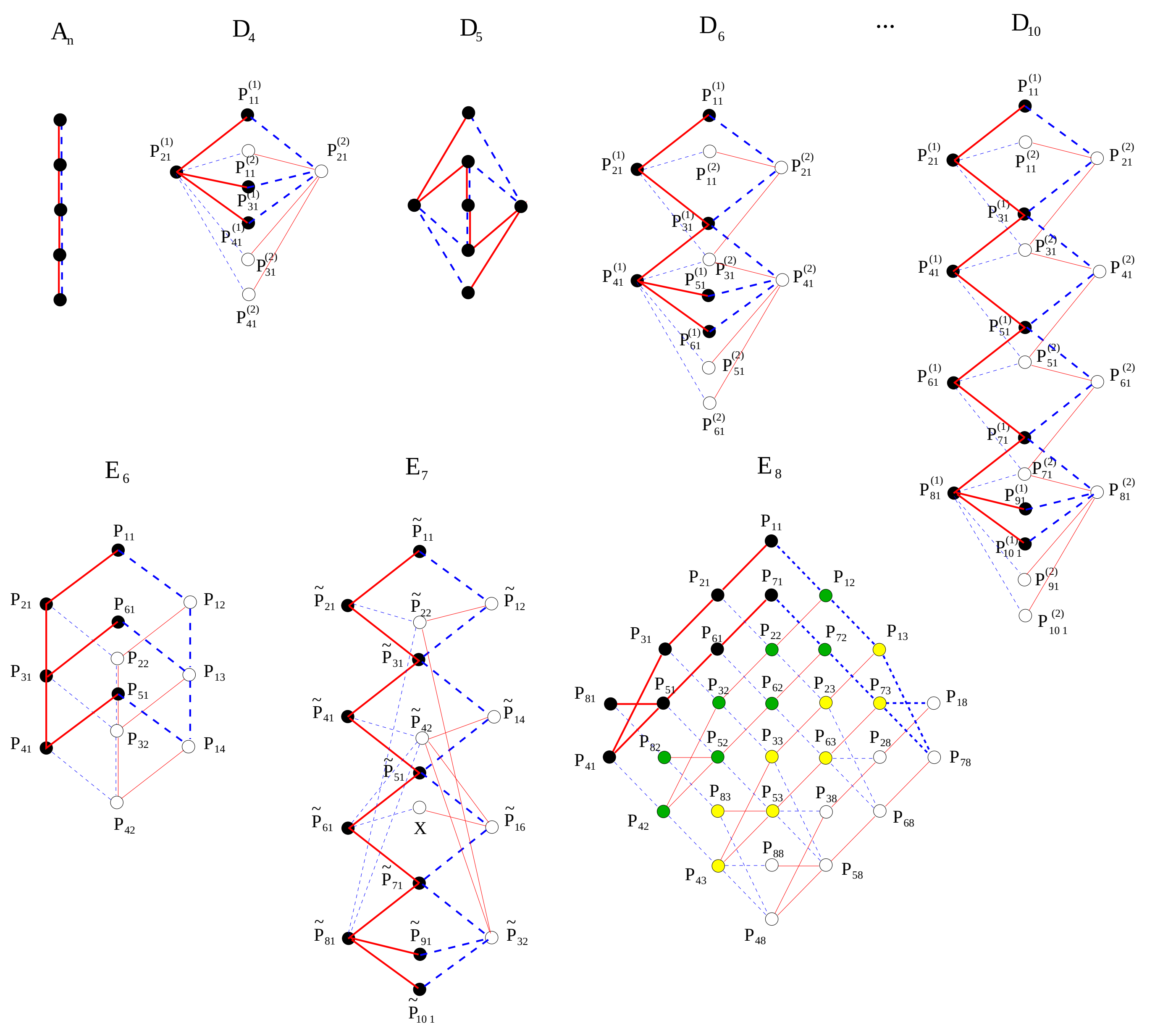
$E_5^{(12)} =$ 

$E^{(24)} =$ 

More algebraic features : graph algebra. . .Operator algebra interpretation in the language of bi-modules [Ocneanu] or of sectors [Xu, Böckenhauer-Evans-Kawahigashi]. . .

Graphs of $\widetilde{V}$? Ocneanu graphs !

For $sl(2)$, two generators $\widetilde{V}_{21}$: red full lines and $\widetilde{V}_{12}$ blue broken lines



A_n
D_4
D_5
D_6
...
D_10
E_6
E_7
E_8
X

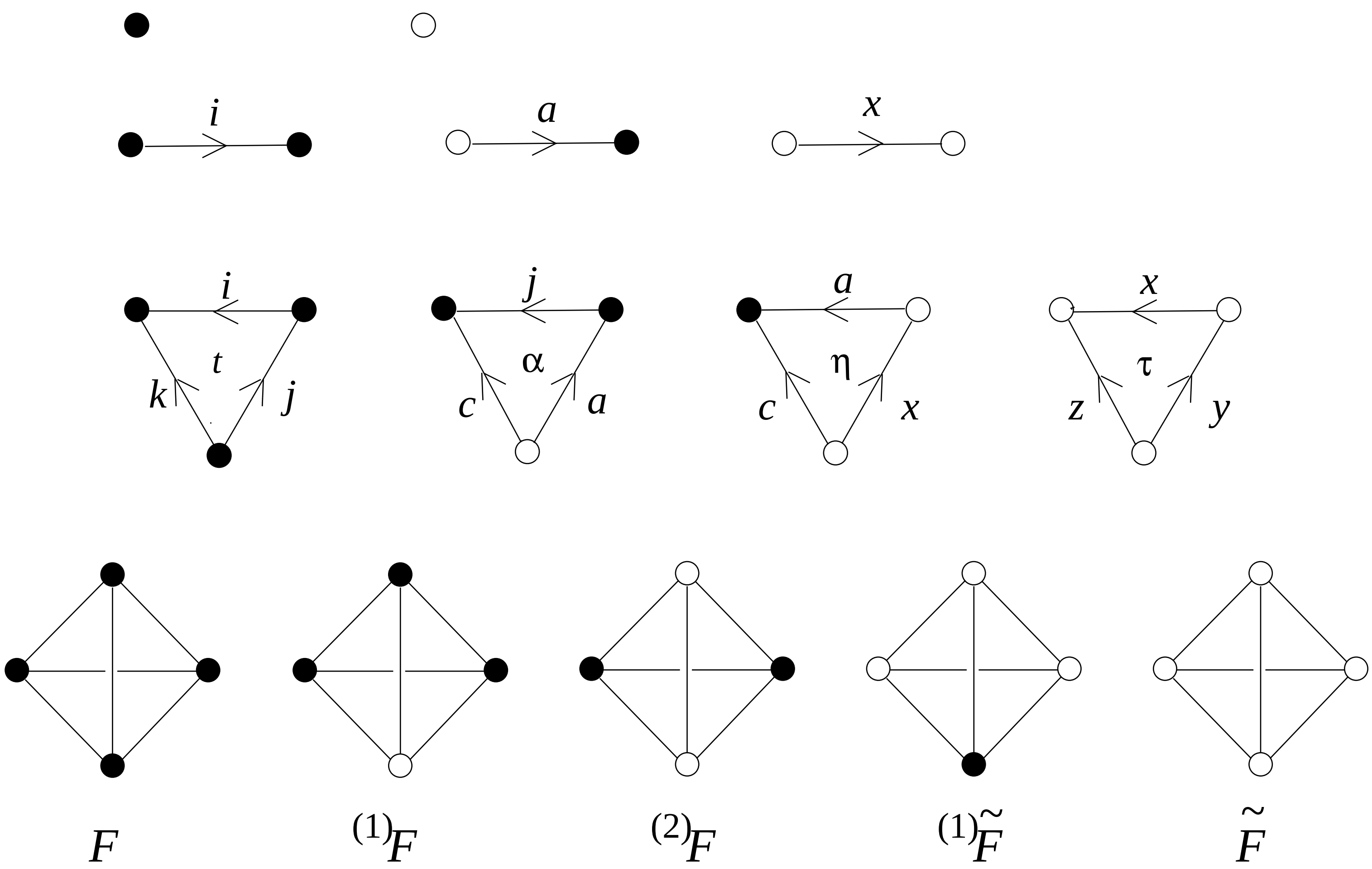
From Nimreps or Graphs to Cells

These multiplicities (or graphs) specify the spectrum of the RCFT in various situations.

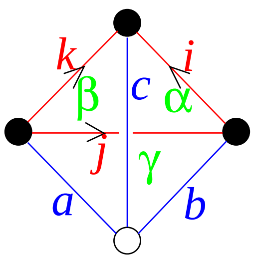
Beyond the spectrum, (Field) “Operator Product Algebra” requires to attach other quantities, cells, to these graphs.

Together, these data enable one to construct a new quantum algebra A (and its dual $\widetilde{A}$) : Quantum groupoid [A. Ocneanu], in fact a Weak Hopf C^* Algebra [Böhm & Szlachanyi].

Construct a simplicial 3-complex, with $\mathbb{C}$-valued 3-cochains or "cells"



For example



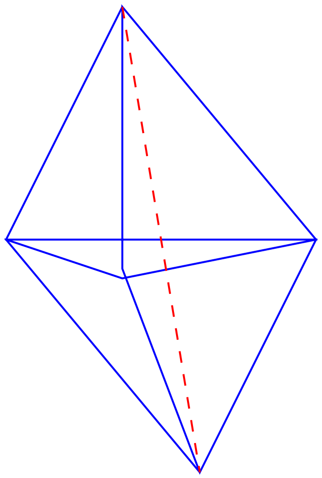
$$= {}^{(1)}F_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta t}_{\alpha\gamma}, \; t = 1, \cdots, N_{ij}{}^{k}, \; \alpha = 1, \cdots, n_{ib}{}^{c} \text{ etc}$$

Square matrices by virtue of nimrep relations (*)

$$\sum_{b,\alpha,\gamma} 1 = \sum_{b} n_{ja}{}^{b} n_{ib}{}^{c} = \sum_{k} N_{ji}{}^{k} n_{ka}{}^{c} = \sum_{k,\beta,t} 1$$

"Big Pentagon": set of 6 quintic identities relating the ${}^{(i)}F$.

$$FF = FFF \ , \qquad F\,{}^{(1)}F\,{}^{(1)}F = {}^{(1)}F\,{}^{(1)}F, \quad \text{etc}\cdots$$



Also unitarity of cells for $F, {}^{(1)}F, {}^{(1)}\widetilde{F}, \widetilde{F}$.

Physically, cells ${}^{(1)}F$ give

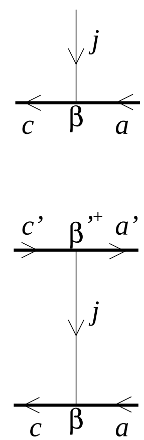
- OPA coefficients in the bulk and on the boundary [Runkel, P.-Z.]
- Boltzmann weights of associated lattice models solutions of Yang-Baxter equation [Pasquier, Ocneanu, P.-Z.]

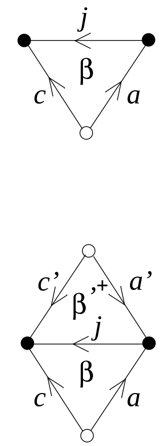
¿ Physical rôle of ${}^{(2)}F, {}^{(1)}\widetilde{F}, \widetilde{F}$?

Refined problem : Find nimreps consistent with these pentagon identities ??

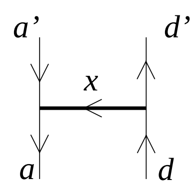
The Double Triangle Algebra

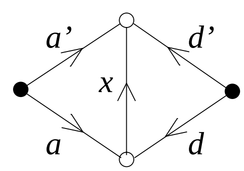
Vector space V^j of dim $\sum n_{ja}{}^c$ with orthogonal basis $|e_{ca}^{j,\beta}\rangle \quad \beta = 1,\cdots,n_{ja}{}^c$
Algebra $A = \oplus_{j\in J}\,\mathrm{End}\,V^j$ generated by "double triangles" $e_{j;\beta,\beta'}^{(ca),(c'a')} = \text{const.}\ |e_{ca}^{j,\beta}\rangle\langle e_{c'a'}^{j,\beta'}|$





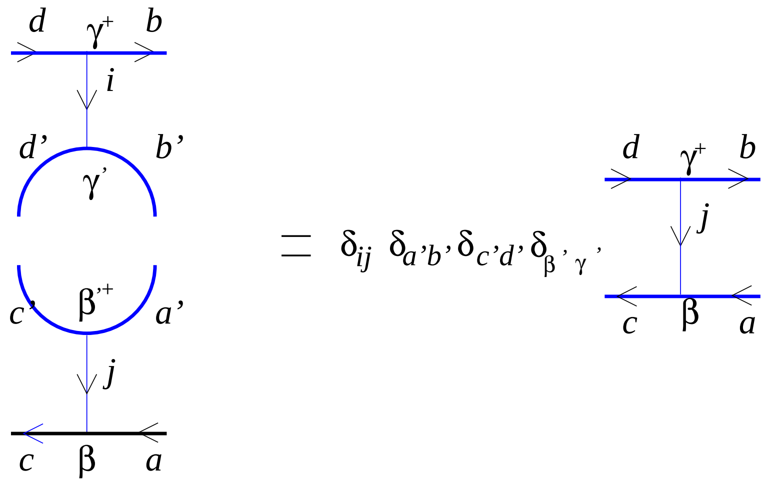
Dual algebra $\widehat{A}$ generated by

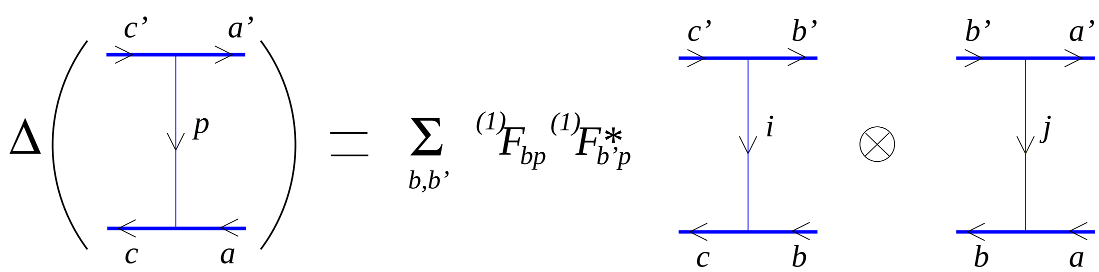




and $\dim(A) = \sum_i (\sum_{a,b} n_{ia}{}^b)^2 = \sum_x (\sum_{a,b} \tilde{n}_{ax}{}^b)^2 = \dim(\widehat{A})$

Product and coproduct





Coassociative coproduct, $\exists$ counit ε, but not a homomorphism and $\Delta(1) \neq 1 \otimes 1$: Weak C^* Hopf Algebra.

This quantum algebra

- is based on combinatorial data (multiplicities) of the RCFT;
  F, ${}^{(1)}F$ are its $6j$-, resp. $3j$- (recombination) symbols etc
- incorporates in a natural way the truncation of repns inherent to RCFT
- enables one to construct Generalized Chiral Vertex Operators, appropriate to describe bulk-boundary coupling

The natural quantum algebra of the RCFT!

Perspectives

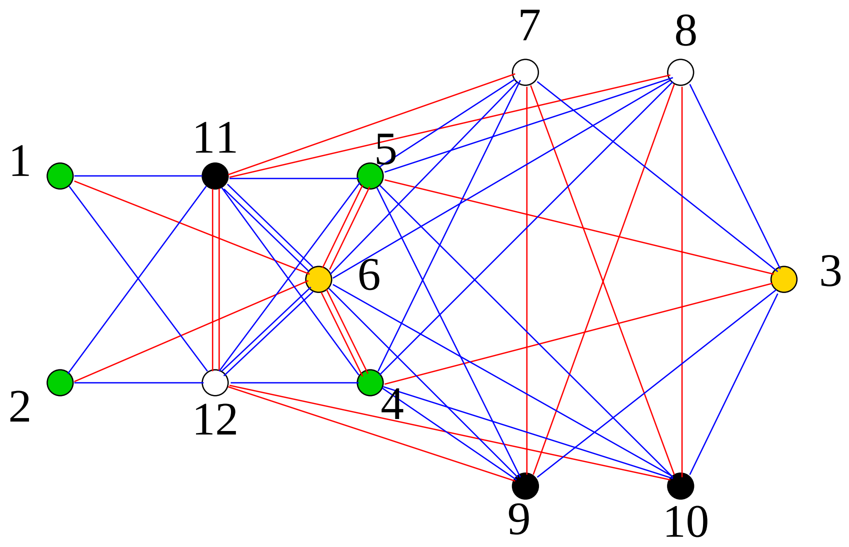
⋆ Need of general results : systematic construction of nimreps, existence/unicity of cells,…

⋆ Loose points : geometrical interpretation of these graphs/nimreps? Generalized root systems? Generalized McKay correspondence? Tensor product multiplicities?…

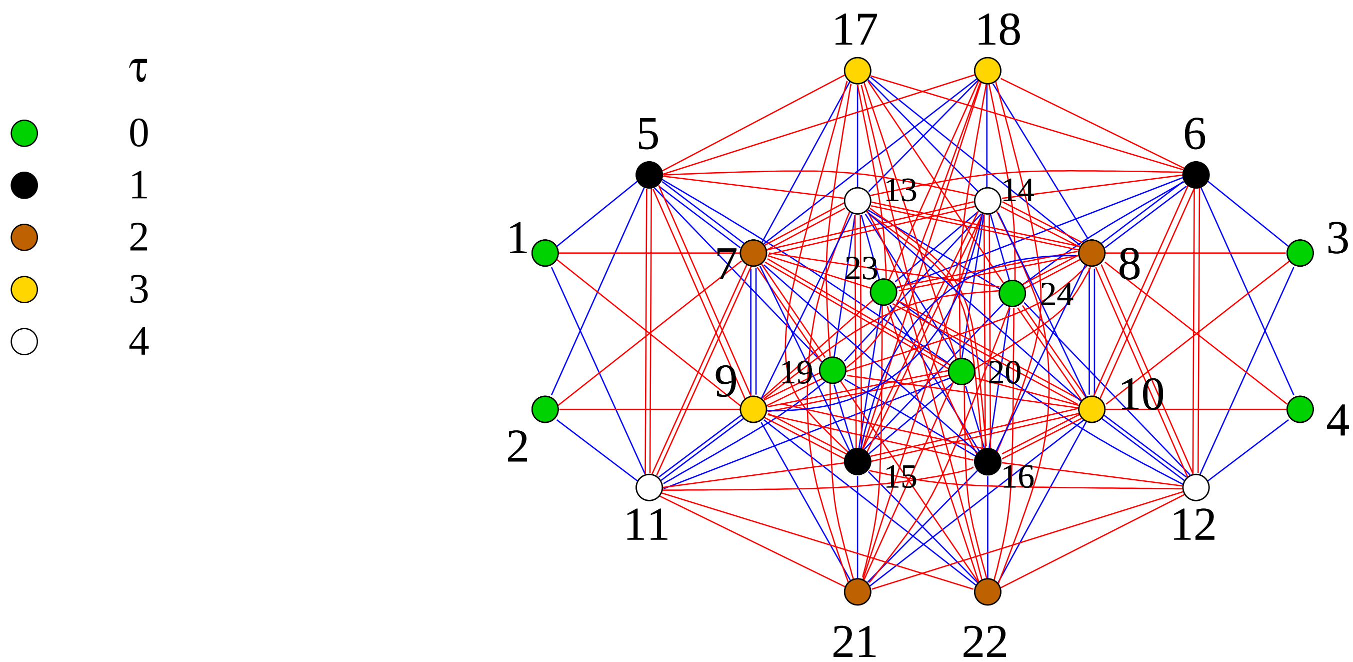
⋆ Extend these considerations to non rational (for example $c \geq 1$) or non compact (Liouville, $SL(2,\mathbb{R})$) : continuous spectrum !

Appendices : further nimreps

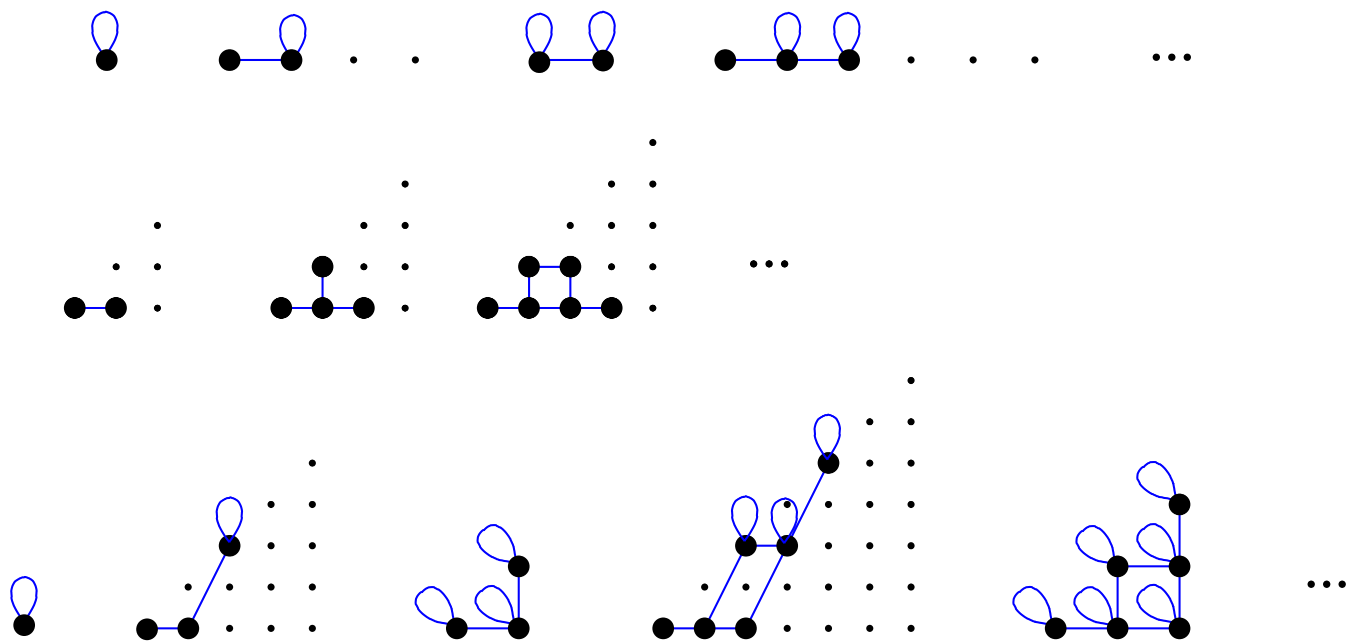
A sample of $\widehat{sl}(4)$ and $\widehat{sl}(5)$ exceptional graphs



$\widehat{sl}(4)_4$



$\widehat{sl}(5)_5$



The $A^{(*)}$ graphs of $\widehat{sl}(N)$, $N = 3, 4, 5$