Exploring a moduli space
Conformal field theory on K3

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Standard conjecture : For $d \le 11$ there is a moduli space of supersymmetric M theories (vacua) with d flat spacetime direction.

81 dimensional component for $d = 6$:

IIA on K3 | $\xrightarrow{\text{change of string coupling}}$ | heterotic string on 4-dim tori $(4_2, 20_L)$

80 dimensional moduli spaces of CFTs as boundaries, well understood at the heterotic side.

Problems : – prove the conjecture
– describe the CFTs on K3 (partition functions, OPE)
behaviour at conifold singularities,
In particular this would yield an explicit description of the Ricci flat metrics on K3.

Seiberg
Aspinwall, Morrison
Ramgoolam, Waldram
Brunner, Entin, Römelsberger
WN, Wendland
Gukov, Vafa

Nikulin
Mukai
Todorov

How to determine the metric on a Riemannian manifold (orbifold) X :

1. attempt : quantum mechanics of free particle on X.

$\Rightarrow \mathcal{L}^2(X)$ graded by eigenvalues of $-\Delta$, partition function.

But in general one cannot hear the shape of a drum.

However, the multiplicative structure on the graded Hilbert space $\mathcal{L}^2(X)$ should determine the metric.

2. atempt : string theory at vanishing coupling, i.e. conformal field theory (CFT)

For large radius R of X, the space $\mathcal{L}^2(X)$ can be approximated by a subspace of the CFT Hilbert space (particle like states of the string).

String Hilbert spaces have a multiplicative structure (operator product expansion, OPE). They are graded by the conformal dimensions (h_L, h_R), $h_L - h_R \in \mathbb{Z}$.

$$\text{OPE} \xrightarrow[R\to\infty]{} \text{multiplication on } \mathcal{L}^2(X)$$

$$h_L = h_R \sim 1/R^2 \longrightarrow \text{eigenvalue of } -\Delta$$

For diagonal rational CFTs (see Zuber's talk) one has a canonical finite dimensional subspace of $h_L = h_R$ states (ground states of W algebra).

Basis f_k, $k = 1, \ldots, N$

Their spectrum yields 'points' x_i, $i = 1, \ldots, N$ (D_0 branes).

Normalize f_k by $\max_i |f_k(x_i)| = 1$.

A distance of the x_i is given by

$$\| x_i - x_j \| = N \max_k |f_k(x_i) - f_k(x_j)| / \sqrt{h_L(f_k)}$$

This idea is modelled after the torus case:

Circle: $\quad f_k f_{k'} = f_{k+k'} \qquad k \in \mathbb{Z}/N\mathbb{Z}$

$$f_k = \exp\left(\frac{2\pi i k \ell}{N}\right) \quad \text{points } \ell = 1, \ldots, N$$

$$|f_k(\ell_1) - f_k(\ell_2)| \sim |k| \, |\ell_1 - \ell_2| / N \quad \text{for small } |k|$$

$$h_L(f_k) \sim k^2 \qquad \text{"}$$

Closely related to "fuzzy geometry", Riemannian non-commutative geometry (Connes)

$\Rightarrow$ the components of the moduli space of CFTs with extended supersymmetry are smooth Riemannian orbifolds.

Thus they have an orbifold fundamental group (Thurston), the "duality group"

$$\mathcal{M} = \frac{\text{simply connected Riemannian manifold}}{\text{duality group}}$$

By definition, the moduli space of CFTs comes with fibres $V(S^1)$, which carry representations of the (extended) Virasoro algebra. For extended susy, some of the irreps cannot be deformed

$\Longrightarrow$ Witten genus of CFT.

For $c_L = c_R = 6$ (and spectral flow $\in V(S^1)$) the axioms imply that the Witten genus is either the elliptic genus of K3 or 0 (= elliptic genus of T^4).

Without the spectral flow assumption, Enriquez surfaces should be possible, too (needs study).

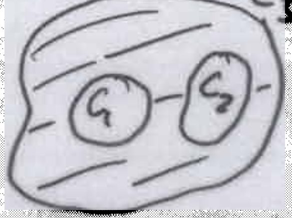
Components $\underset{\text{connected}}{\underline{\mathcal{M}(T^4)}}$, $\mathcal{M}(K3)$ of CFT moduli space.

Recall Segal's CFT axioms:

Category of Riemann surfaces with (incoming and outgoing) boundary, central extension c_L, c_R (mostly $c_L = c_R$).

Functor to category of linear maps between vector spaces.

Ex.



$$V(C_1) \otimes V(C_2) \to V(C_3)$$

Cylinders yield action of $Vir_L \times Vir_R$ on $V(C)$, in particular grading by h_L, h_R.

Supersymmetric extensions exist.

One can define a topology on the set of CFTs.

Standard conjecture:

All deformations are given by $(h_L, h_R) = (1,1)$ states.

For CFTs with $N=2$ supersymmetry, all $(\frac{1}{2}, \frac{1}{2})$ fields with correct charges yield deformations. (proved by Dixon in perturbation theory)

Note the analogy to complex deformations of Calabi-Yau manifolds.

standard conjecture:
For large radius R there is a local isomorphism of moduli spaces

supersymmetric σ-models on Calabi-Yau manifolds X $\longrightarrow$ CFTs with extended supersymmetry

$H^*(X,\mathbb{R})$ $\longrightarrow$ non-deformable irreps in $V(S^n)$
real subspace $\bigoplus_i \mathcal{H}_i$

Holonomy group of moduli space $\longrightarrow$ $\times_i SO(\mathcal{H}_i)$

For independent $\mathcal{H}_i$ this implies that the moduli space $\mathcal{M}$ is a local product '$\mathcal{M}_{complex} \times \mathcal{M}_{K\ddot{a}hler}$'.

$\mathcal{H}_a \simeq H^{1,1}(X)$, $\mathcal{H}_b = H^{1,n-1}(X)$

n = complex dimension of X.

Hyperkähler spaces
$H^{1,1}(X) \simeq H^{1,n-1}(X)$

(obvious for $n=2$: K3 or T^4)

$\Rightarrow$ no product

σ-models on K3 $\longrightarrow$ moduli space of CFTs
(conjecture by Aspinwall, Morrison, certainly true)

R=0 metrics / scale $\times$ B-field $\times$ volume $\longrightarrow$ moduli space

classified by $\underbrace{H^2(K3,\mathbb{Z})}_{\text{even selfdual lattice}} \subset H^2(K3,\mathbb{R}) \simeq \mathbb{R}^{3,19}$

and action of Hodge $*$ on $H^2(K3,\mathbb{R})$

$*^2 = 1$ $+1$ eigenspace : pos. def. 3-plane in $H^2(K3,\mathbb{R})$
-1 "   neg. def. 19-plane   "

Aspinwall and Morrison constructed a natural map from the σ-model data to the Grassmannian of positive definite 4-planes in $H^*(K3,\mathbb{R}) \supset H^*(K3,\mathbb{Z})$
$\simeq \mathbb{R}^{4,20}$

$H^*(K3) = H^0(K3) \oplus H^2(K3) \oplus H^4(K3)$
dimensions   1   22   1

Take intersection of 4-plane and orthogonal 20-plane with $H^4(K3)^\perp$, project to $H^2(K3)$:

3-plane and 19-plane in $H^2(K3)$,
projection given by 3+19 components of B-field $\in H^2(K3)$
4^{th} vector in 4-plane $\longrightarrow$ volume

Dualities: Forget the marking of $H^0(K3)$ and $H^4(K3)$ in $H^*(K3)$.

Proved under standard assumptions

Orbifolding:

Let $\mathcal{M}_G$ be the subspace of theories in $\mathcal{M}(T^4)$ which are invariant under a finite group G $\Rightarrow$ new theories for which

$$\mathcal{H} = \mathcal{H}(T^4)^G \oplus \mathcal{H}^{twisted}$$

In many cases the new theories belong to $\mathcal{M}(K3)$.

Ex. Kummer surfaces, $G = \mathbb{Z}_2$

$$\mathcal{H}(T^4) = \underbrace{Fock(j_L, j_R)}_{\text{differential polynomials in } j_L, j_R \in \mathbb{R}^4} \otimes \mathbb{C}[\Lambda]$$

Λ even self-dual lattice in $\mathbb{R}^{4,4}$

$\mathbb{Z}_2$ action induced by $j_R^L \longmapsto -j_R^L$, $\lambda \longmapsto -\lambda$ for $\lambda \in \Lambda$

possible for all of $\mathcal{M}(T^4)$.

Classically:

$T^4 = \mathbb{R}^4/L$ $\quad$ L: latice of rank 4

$\mathbb{Z}_2$ action on T^4: $\quad x \longmapsto -x$

2^4 fixpoints $\quad \lambda \in L/2 \bmod L$.

$$H^2(\widetilde{T^4/\mathbb{Z}_2}, \mathbb{Z}) \supset \Pi = H^2(T^4, \mathbb{Z})[2]$$

$$\cup$$

$$K = \langle E_\lambda \mid \lambda \in L/2 \bmod L \rangle$$

$$\oplus \left\langle \tfrac{1}{2} \sum_{\lambda \in \mathcal{C}} E_\lambda \;\middle|\; \mathcal{C} \text{ hyperplane in } L/2 \bmod L \right\rangle$$

$\Pi \oplus K$ has a unique extension to a selfdual lattice, since

$$\Pi^*/\Pi \simeq K^*/K \quad (\simeq \mathbb{Z}_2^6) \qquad \text{(Nikulin)}.$$

The corresponding procedure for the CFT case needs a modification:

$$H^*(\widetilde{T^4/\mathbb{Z}_2}) \supset \tilde{\Pi} = H^{even}(T^4, \mathbb{Z}_2)[2]$$

$$\cup$$

$$K \quad \text{(same as before)}$$

But now

$$\tilde{\Pi}^*/\tilde{\Pi} \simeq \mathbb{Z}_2^8 \neq \mathbb{Z}_2^6 \simeq K^*/K.$$

$\tilde{\Pi} \oplus K$ cannot be extended to a selfdual lattice $\Rightarrow$ non-zero B-field through some elements of K.

For $\tilde{K} = \tilde{\Pi}^\perp \cap K$ one needs $K/\tilde{K} \simeq \mathbb{Z}_2$.

$\tilde{K}^*/\tilde{K} \simeq \mathbb{Z}_2^8$ thus

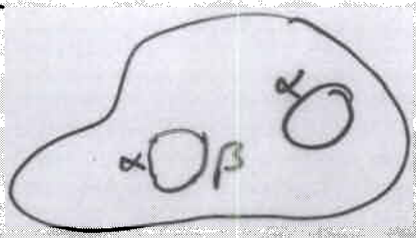
Unique solution compatible with symmetries: B field $1/2$ through all E_i.

The procedure can be generalised to all possible $G \subset SU(2)$ and always yields unique solutions.

The results can be interpreted in terms of boundary states (~D branes)

Axioms by Segal and Moore, extend Segal's CFT axioms.

Segments of boundary can be labelled by index set



$\alpha \in V(S^1)$

Projection to non-deformable subspace yields lattice point x_α.

For K3: states with $h_L = h_R = \frac{1}{4}$ (RR ground states), charged ⊕ uncharged $= \mathbb{R}^4 \oplus \mathbb{R}^{20}$

yields intersection form (integral on lattice), pos. def. on $\mathbb{R}^4$, neg. def. on $\mathbb{R}^{20}$.

Lattice + pos. def. 4-plane $\Longrightarrow$ point in $\mathcal{M}(K3)$

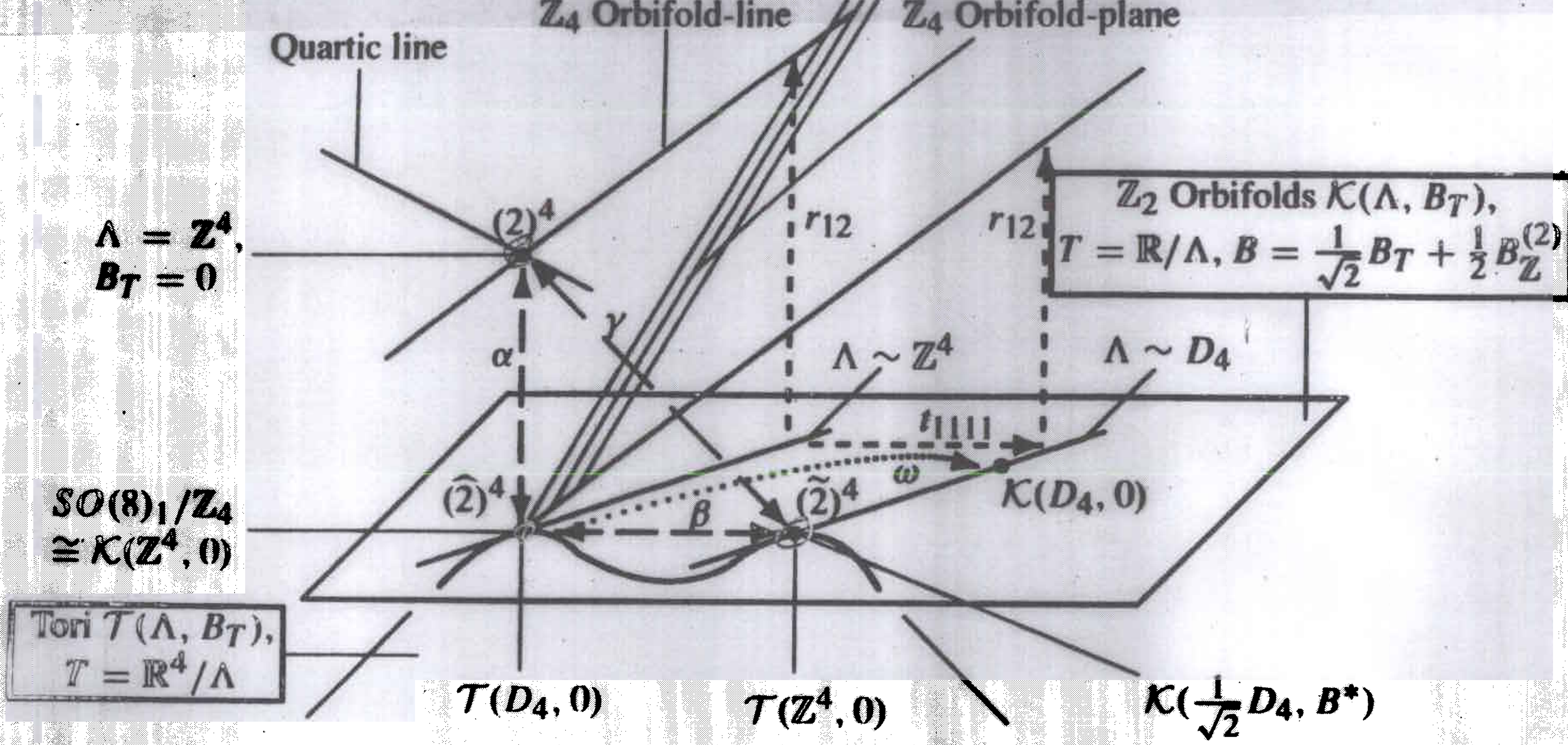


Fig. 4.1. Strata of the moduli space

This should allow the placing of Gepner points in $\mathcal{M}(K3)$ (Recknagel, Schomerus)

$$\mathcal{M}(T^4) = {}_{O(4)\times O(4)}\backslash O(4,4)/_{O(\text{lattice})}$$

$$\mathcal{M}(K3) = {}_{SO(4)\times O(20)}\backslash O^+(4,20)/_{O^+(\text{lattice})}$$

Shimura varieties with CM points (pos. def. 4-planes spanned by lattice vectors)

Conjecture (Kontsevich, Nahm, Gukov+Vafa):

CM points = rational theories

True for $\mathcal{M}(T^4)$ and its orbifolds, Gepner points.

Question: What are the W-algebras for the CM points in $\mathcal{M}(K3)$?

What about other hyperkähler varieties?

Other Calabi-Yau spaces?