Quantum Liouville theory versus Quantization of Teichmüller space

Conjecture (H. Verlinde)

One has a canonical isomorphism

$$(\mathcal{H}^{L}_{g,n}, \pi^{L}_{g,n}) \simeq (\mathcal{H}^{T}_{g,n}, \pi^{T}_{g,n}),$$

where:

$\mathcal{H}^{L}_{g,n}$: space of conf. blocks in Liouville theory

$\mathcal{H}^{T}_{g,n}$: Space of states obtained by quantizing Teichmüller spaces of Riemann surfaces,

$\pi^{T}_{g,n}, \pi^{L}_{g,n}$: the corresponding reprs. of the mapping class group.

Motivation:

Wish to go beyond RCFT.

Liouville Theory

- The noncompact minimal model based on the following series of unitary hwr:

$$\{ \mathcal{V}_\Delta \, ; \, \Delta \in (\tfrac{c-1}{24}, \infty) \}$$

"Principal series of unit. hwr of Vir"

$c > 1$: Central charge of Vir

Definition and analysis of fusion of these representations may be approached with the help of a calculus of chiral vertex operators:

$$\Psi^{\Delta_2}_{\Delta_3 \Delta_1}(z) : \mathcal{V}_{\Delta_1} \to \mathcal{V}_{\Delta_3} \quad \text{s.t.}$$

$$L_n \Psi^{\Delta_2}_{\Delta_3 \Delta_1}(z) - \Psi^{\Delta_2}_{\Delta_3 \Delta_1}(z) L_n =$$
$$= z^n (z\partial + \Delta_2 (n+1)) \Psi^{\Delta_2}_{\Delta_3 \Delta_1}(z)$$

Such operators can be constructed for any triple $(\mathcal{V}_{\Delta_3}, \mathcal{V}_{\Delta_2}, \mathcal{V}_{\Delta_1})$ of reprs. from the Virasoro principal series.

(Free field constr. involving complex powers of screening charges)

Useful param.: $\Delta = \Delta_\alpha = \alpha(Q-\alpha)$, $Q^2 = \frac{c-1}{6}$

Key Technical result : braid relations

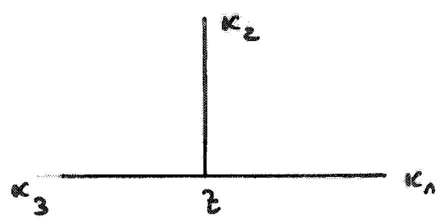
$$V_{\alpha_4\ \alpha}^{\ \alpha_3}(z_2)\, V_{\alpha\ \alpha_1}^{\ \alpha_2}(z_1) = \int_{\mathbb{S}} d\alpha'\; B^{\varepsilon}_{\alpha\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} V_{\alpha_4\ \alpha'}^{\ \alpha_2}(z_1)\, V_{\alpha'\ \alpha_1}^{\ \alpha_3}(z_2),$$

where :

- $\varepsilon = \mathrm{sgn}(\sigma_2 - \sigma_1)$ if $z_2 = e^{i\sigma_2}$, $z_1 = e^{i\sigma_1}$
- $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$
- Coeffs. $B^{\varepsilon}_{\alpha\alpha'}$: explicitly known (sort of 6j's)

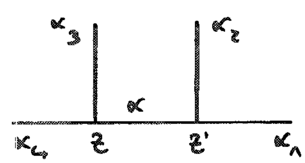
Genus 0 : 1.

This result forms the basis for a description of conf. blocks and their monodromies à la Moore and Seiberg:

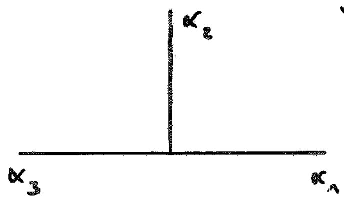


$$\hat{=} \quad V^{\alpha_2}_{\alpha_3\alpha_1}(z)$$

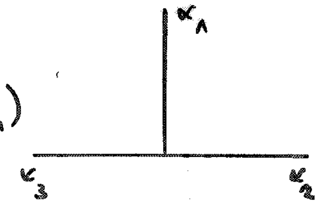
Spaces of conf. blocks from compositions of chiral vertex ops. + bilinear form, e.g.

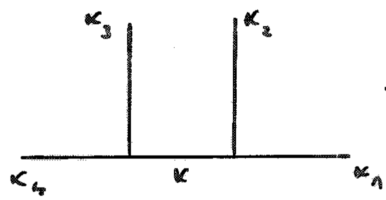
$\mathcal{H}_{0,4} = \mathrm{Span}\Big\{$  $;\ \alpha \in \mathbb{S} \Big\}$

Description of corresponding braid group representation in terms of elementary fusion and braiding moves:



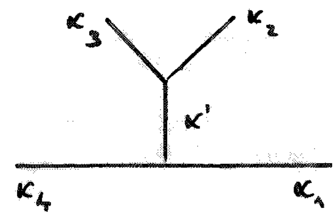
$\longrightarrow \Omega^{L}(\alpha_3, \alpha_2, \alpha_1)$





$\longrightarrow$

$$\longrightarrow \int_{\mathbb{S}} d\alpha' \, F^{L}_{\alpha\alpha'}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}$$



Remark

More satisfactory would be to use Wassermann's notion of fusion (unitarity manifest?). In this framework one would obtain the decomp. of the fusion product $\mathcal{V}_{\alpha_2} \boxtimes \mathcal{V}_{\alpha_1}$ into irreducibles from a particular case of our braid-rels.:

$$\boxed{\mathcal{V}_{\alpha_2} \boxtimes \mathcal{V}_{\alpha_1} \simeq \int_{\mathbb{S}} d\alpha \;\; \mathcal{V}_{\alpha}}$$

work left to be done

$\rightsquigarrow \dots \rightsquigarrow$ Principal series of Vir-hwr forms braided tensor category.

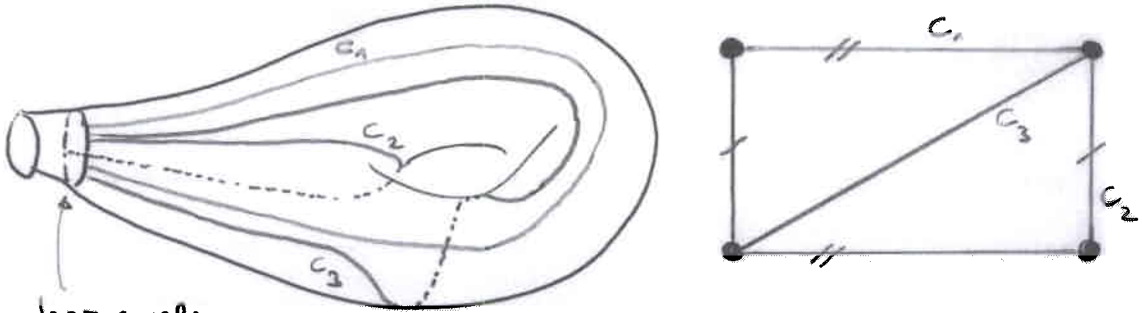
Kashaev;

Chechov, Fock

Quantization of Teichmüller space

Ⓐ Coordinates on $\mathcal{T}_{g,n}$, (Penner) $n \geq 1$

Example:



Decorated Teichmüller space $\tilde{\mathcal{T}}_{g,n}$:
Fiber sp. over $\mathcal{T}_{g,n}$ with fiber: n-Tuple of horocycles about holes of $\Sigma_{g,n}$

- Hyperbolic Lengths of geodesics isotopic to c_1, c_2, c_3 : coordinates on $\tilde{\mathcal{T}}_{1,1}$.
- Ratios of lengths $\frac{\lambda(c_i)}{\lambda(c_j)}$ represent coordinates for $\mathcal{T}_{1,1}$.

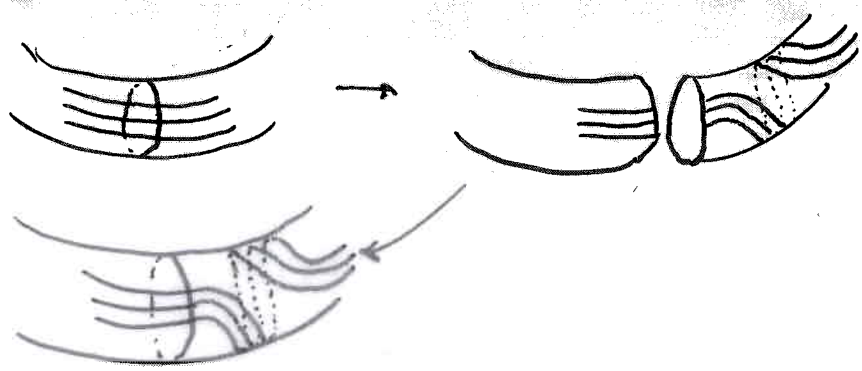
The (decorated) Teichmüller space is symplectic: The symplectic str. associated to the (pullback of the) Weil-Petersson Two-form Takes a simple expression:

Penner, Wolpert

$$\omega_{WP} = 4\left(d\log\lambda_1 \wedge d\log\lambda_2 + d\log\lambda_2 \wedge d\log\lambda_3 + d\log\lambda_3 \wedge d\log\lambda_1 \right)$$

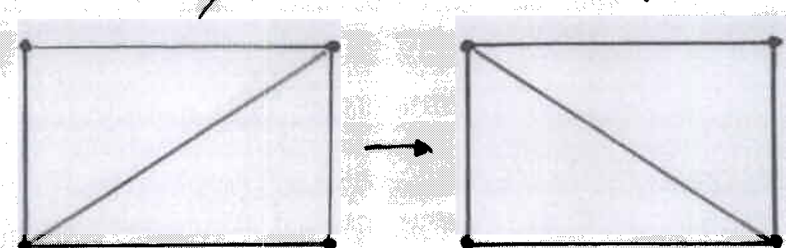
Ⓑ Action of the mapping class group*)

*) generated by Dehn-Twists:



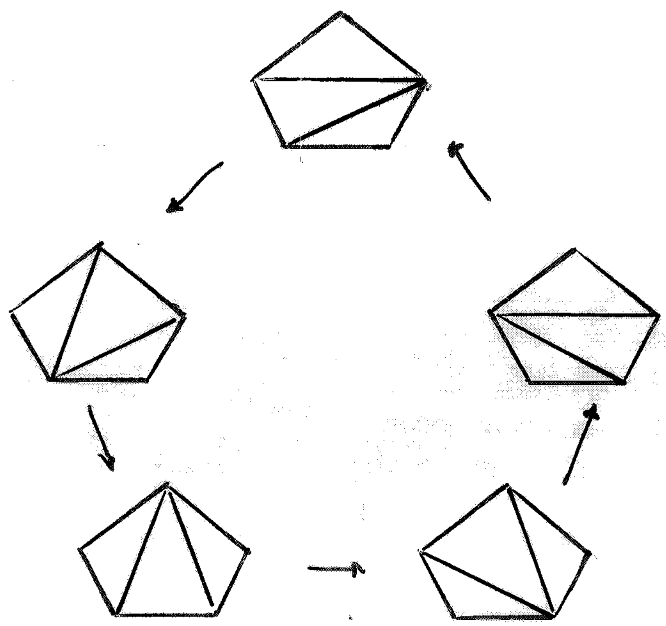
An element of the mapping class group will change the Triangulation drawn on a given R.S.

The point is that the resulting action can be "undone" by a sequence of elementary moves ("flips")

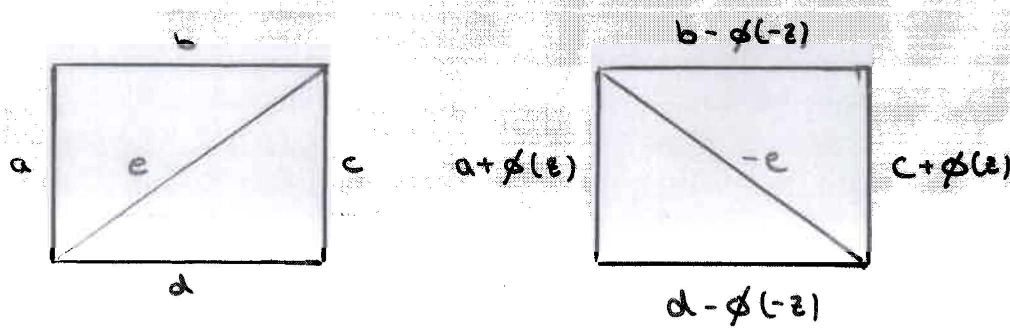


In other words: The mapping class group imbeds into the "Ptolemy groupoid" generated by the flips.

The Ptolemy groupoid has (unlike the mapping class group) a simple description in terms of generators (flips) and relations!



Transformation of the (Fock-)*) coordinates:



$$\phi(z) := \ln(e^z + 1)$$

*) Essentially cross-ratio of Penner-coordinates.

Quantization

- The simple part -

Introduce operators $\hat{z}_i$ for each variable z_i associated to an edge in the triangulation

Relations:

$$[\hat{z}_i, \hat{z}_j] = 2\pi i \hbar \{z_i, z_j\} \qquad \hat{z}_i^\dagger = \hat{z}_i$$

Call the resulting algebra $\mathcal{A}_{g,n}$. Note that the center of $\mathcal{A}_{g,n}$ is generated by elements "measuring the geodesic length of the boundary holes". Assign numerical values to each of these generators.

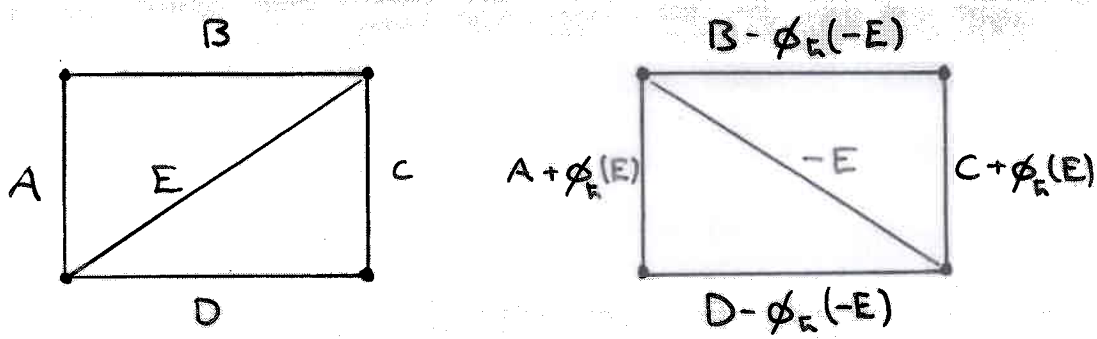
Fock

Then choose a maximal commuting subset of the remaining generators, represent them as multiplication operators (Schrödinger-representation).

$$\rightsquigarrow \mathcal{H}^T_{g,n} \simeq L^2(\mathbb{R}^{3g-3+n})$$

Quantization, interesting part:

– Representation of the Ptolemy groupoid –



$$\phi_\hbar(z) := -\frac{\pi\hbar}{2}\int d\tau \, \frac{e^{-i\tau z}}{\sinh\pi\tau \, \sinh\pi\hbar\tau}$$

$[E, A] = 2\pi i\hbar$ $[E, C] = 2\pi i\hbar$

$[B, E] = 2\pi i\hbar$ $[D, E] = 2\pi i\hbar$

Remarks

1) There exist unitary operators that represent these moves in $\mathcal{H}_{g,n}$

2) Resulting representation of the mapping class group is projective with factor

$$\xi = e^{\pi i c_h}, \quad c_h = 1 + 6(b+b^{-1})^2$$

$$b = \sqrt{\hbar}$$

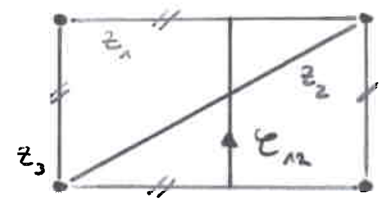
Kashaev

Geodesic length operators

The work of Fock also contains a prescription for the construction of operators measuring the geodesic length of a cl. curve.

Example:

$\Sigma_{1,1}$



Variables : z_1, z_2, z_3

Relations : $[z_1, z_2] = 4\pi i \hbar$ and cyclic

Central element : $z_1 + z_2 + z_3 \equiv c$

$\Rightarrow \quad \mathcal{H}_{1,1} \simeq L^2(\mathbb{R})$

Geodesic length operator associated to e_{12}:

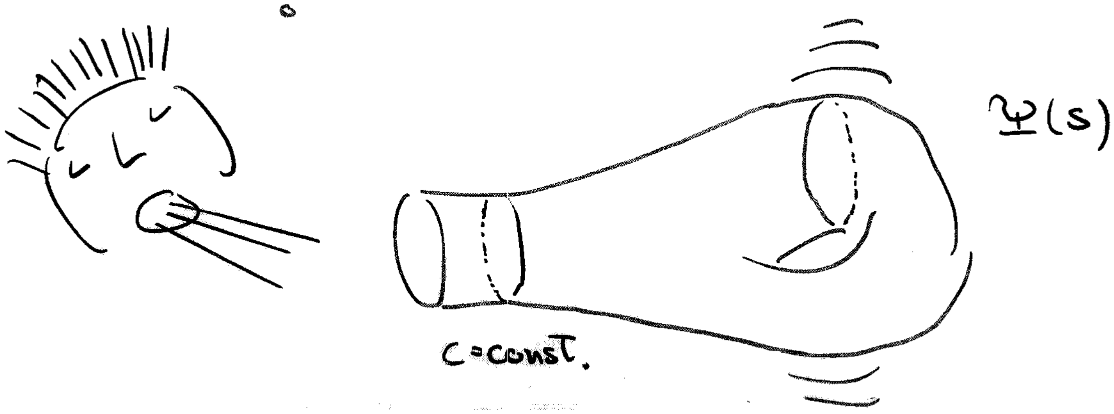
$$L_{12} = 2\cosh(2\pi b p) + e^{2\pi b x}$$

$$2\pi b p = \tfrac{1}{2}(z_1 + z_2), \qquad 2\pi b x = \tfrac{1}{2}(z_1 - z_2), \qquad b = \sqrt{\hbar}$$

L_{12} is self-adjoint, eigenstates $|s\rangle$

$$L_{12}\,|s\rangle = 2\cosh(2\pi b s)\,|s\rangle$$

$$\int_0^\infty ds\; |s\rangle\langle s| = \mathbb{1} .$$

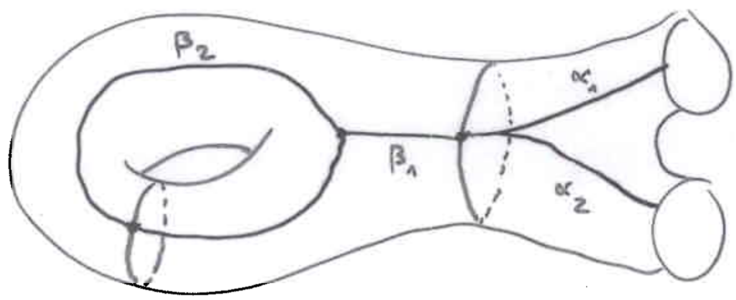


Representations of the mapping class group from Moore-Seiberg data

Let $\mathcal{J}$ be an (index-) set

($\triangleq$ labels for Vir-reprs.)

There is another useful set of graphs drawn on a Riemann-suface to characterize pants-decomp. of the surface:

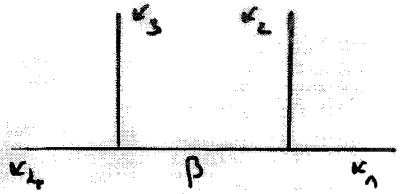


Each edge is assumed to be colored with a label $\in \mathcal{J}$.

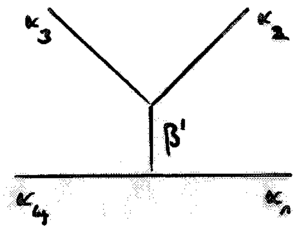
Bases for the associated spaces of conformal blocks are obtained by varying the "internal labels, e.g.

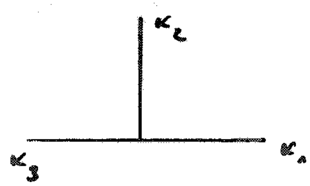
$$\mathcal{H}_{1,2} = \text{Span}\left\{ \bigcirc_{\beta_2}\!\!-\!\!\overset{\beta_1}{}\!\!<^{\alpha_1}_{\alpha_2} \; ; \; \beta_1, \beta_2 \in \mathcal{J} \right\}$$

This yields a linear (ultimately Hilbert-) space that depends on $\Sigma_{g,n}$ only, provided there ex. lin. trsf. betw. the bases to diff. graphs.

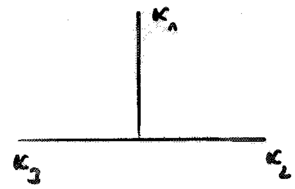


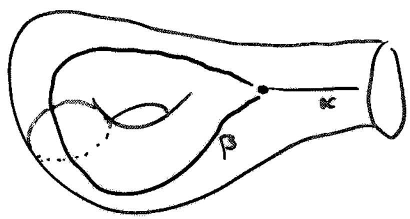
$$\longrightarrow \sum_{\beta' \in J} F_{\beta\beta'} \begin{bmatrix} \kappa_3 & \kappa_2 \\ \kappa_4 & \kappa_1 \end{bmatrix}$$



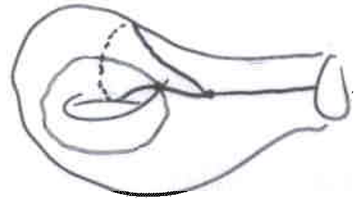


$$\longrightarrow \Omega(\kappa_3, \kappa_2, \kappa_1)$$





$$\longrightarrow \sum_{\beta' \in J} S_{\beta\beta'}(\kappa)$$



Again the game is to generate a groupoid from the above elementary moves subject to certain relations:

<u>Thm</u>: (Moore-Seiberg, Bakalov-Kirillov)

All relations come from $\Sigma_{0,5}$ and $\Sigma_{1,2}$, most importantly:

$$\sum FFF = FF \qquad \text{(Pentagon)}$$

$$\sum F\Omega F = \Omega F \Omega \qquad \text{(Hexagon)}$$

"$SaS^{-1} = b$" - relation

As in the previous case of the Ptolemy-groupoid one has canonically associated a mapping class group representation.

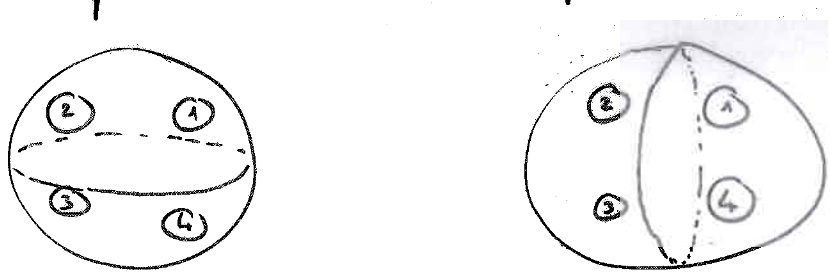
To summarize: We have

- In quantum Teichmüller:
  Assignment $\Sigma_{g,n} \to \mathcal{H}^{T}_{g,n}$, equipped with mapping class group repr. via "flips".
- In quantum Liouville:
  Assignment $\Sigma_{0,n} \to \mathcal{H}^{L}_{0,n}$ (conformal blocks), equipped with mapping class (braid-) group repr. characterized by (F^L, Ω^L).

Relation between the two?

.... can be established by showing that mapping class group repr. from quantum Teichmüller theory is equivalent to a repr. à la Moore - Seiberg

Example: 4-holed sphere, $\mathcal{H}_{0,4} \simeq L^2(\mathbb{R})$



L_{21}, $\Psi^{21}_{s}\left[\begin{smallmatrix} c_3 & c_2 \\ c_4 & c_1 \end{smallmatrix}\right](x)$ $\qquad$ L_{32}, $\Psi^{32}_{s}\left[\begin{smallmatrix} c_3 & c_2 \\ c_4 & c_1 \end{smallmatrix}\right](x)$

Length op. $\qquad$ eigenfct.

$$F^{T}_{ss'}\left[\begin{matrix} c_3 & c_2 \\ c_4 & c_1 \end{matrix}\right] = \int dx \; \overline{\Psi^{32}_{s'}\left[\begin{matrix} c_3 & c_2 \\ c_4 & c_1 \end{matrix}\right](x)} \, \Psi^{21}_{s}\left[\begin{matrix} c_3 & c_2 \\ c_4 & c_1 \end{matrix}\right](x)$$

$$F_{\alpha\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} \propto \begin{Bmatrix}\alpha_1 & \alpha_2 & \alpha\\ \alpha_3 & \alpha_4 & \alpha'\end{Bmatrix},$$

where $\{\ldots\}_b$, the "b-6j symbol" is given as:

$$\begin{Bmatrix}\alpha_1 & \alpha_2 & \alpha\\ \alpha_3 & \alpha_4 & \alpha'\end{Bmatrix}_b = \frac{S_b(\alpha_2+\alpha-\alpha_1)\,S_b(\alpha'+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha'-\alpha_3)\,S_b(\alpha_3+\alpha-\alpha_4)}$$

$$\times \int_{i\mathbb{R}} ds \prod_{i=1}^{4} \frac{S_b(u_i+s)}{S_b(v_i+s)},$$

where:

$$\log S_b(x) = \int_0^\infty \frac{d\tau}{\tau}\left(\frac{\mathrm{sh}(\frac{Q}{2}-x)\tau}{2\,\mathrm{sh}(\frac{b\tau}{2})\,\mathrm{sh}(\frac{\tau}{2b})} - \frac{Q-2x}{\tau}\right)$$

$$u_1 = \alpha+\alpha_1-\alpha_2 \qquad v_1 = 2Q+\alpha-\alpha'-\alpha_2-\alpha_4$$

$$u_2 = Q-\alpha_1+\alpha-\alpha_2 \qquad v_2 = Q+\alpha+\alpha'-\alpha_2-\alpha_4$$

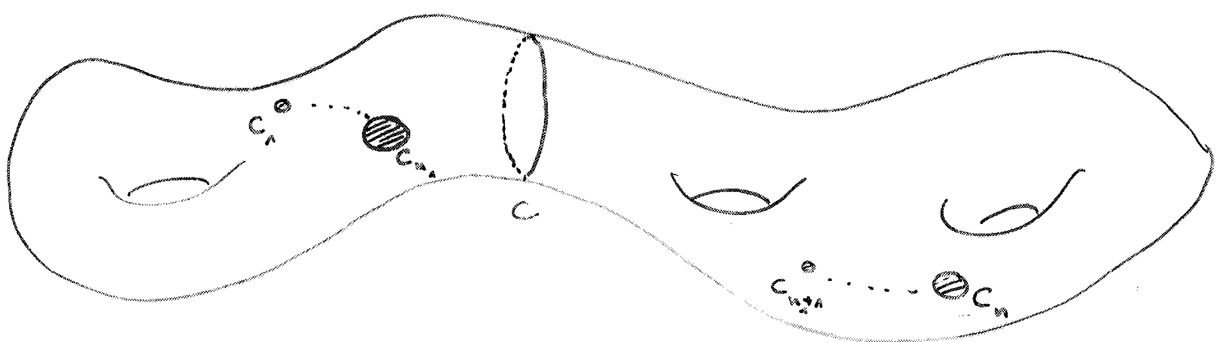
$$u_3 = \alpha+\alpha_3-\alpha_4 \qquad v_3 = 2\alpha$$

$$u_4 = Q-\alpha_3+\alpha-\alpha_4 \qquad v_4 = Q$$

We have verified that:

- The length operators to any noncontractible cycle can be diagonalized. Their diagonalization factorizes $(\mathcal{H}^{T}_{g,n}, \Pi^{T}_{g,n})$:

$$\left(\mathcal{H}^{c_1,\dots,c_n}_{g,n}, \Pi^{c_1,\dots,c_n}_{g,n}\right) \simeq \int_{\mathbb{R}^+} dc \left(\mathcal{H}^{c_1,\dots,c_{n_1};c}_{g_1,n_1}, \Pi^{c_1,\dots,c_{n_1};c}_{g_1,n_1}\right) \otimes \otimes \left(\mathcal{H}^{c;c_{n_1+1},\dots,c_n}_{g_2,n_2}, \Pi^{c;c_{n_1+1},\dots,c_n}_{g_2,n_2}\right)$$



- The mapping class group representation coming from the Ptolemy groupoid can be represented in terms of Moore-Seiberg data (F^T, Ω^T).

  ($S_{c_1 c_2}(c_3)$ can be expressed in terms of (F^T, Ω^T))

- One finds

$$(F^T, \Omega^T) = (F^L, \Omega^L)$$

provided that the parameters c and κ are related as:

$$c = 4\pi b P \quad \text{if} \quad \kappa = \frac{Q}{2} + iP$$

Remark:

There ex. a braided tensor category of reps. of $\mathcal{U}_q(sl(2,\mathbb{R}))$, the "strange" *) series $\mathcal{P}_\alpha$, $\alpha \in \mathbb{S}$, which is characterized by (F^L, Ω^L). ($q = e^{\pi i b^2}$ if $Q = b + b^{-1}$)

One has

$$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathbb{S}} d\alpha \; \mathcal{P}_\alpha$$

*) "strange": No classical limit $b \to 0$, self-duality $b \to b^{-1}$ ("modular double" (Faddeev)

Concluding remarks

The Verlinde conjecture

$$(\mathcal{H}^{T}_{g,n}, \Pi^{T}_{g,n}) \simeq (\mathcal{H}^{L}_{g,n}, \Pi^{T}_{g,n})$$

expresses an amazing link between the (quantum) geometry of Riemann surfaces and the representation theory (harmonic analysis) of Vir (Diff S_1).

We have presented substantial progress towards proving this conjecture (lacking: representation theoretic constr. of conf. blocks in higher genus).

It seems to us that the proper conceptual framework for the mathematical understanding of these connections remains to be found ("quantum universal Teichmüller space").