Conformal Field Theory defined by Regular Chiral Vertex Operator algebras.

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§0. Introduction

I. The purpose of this Talk

Construct "Good" Conformal Field Theory associated with "Good" Vertex operator algebras

II. Two examples

i) Minimal series of Virasoro algebras.

Beilinson - Feigin - Mazur.

ii) Integrable representations of Affine Lie algebras.

Tsu.. - Ueno - Yamada

§1. Chiral Vertex Operator algebras and their modules

I. Def of algebra

$V = (V, J, T)$: chiral vertex operator algebra

$\overset{\text{def.}}{\Longleftrightarrow}$

1) $V = \sum_{\Delta \in \mathbb{Z}_{\geq 0}} V_\Delta$: graded vector space.

· $\dim V_\Delta < \infty$. $V_0 = \mathbb{C}|0\rangle$

· $V_2 \ni T \neq 0$ Virasoro elements.

$$\mathrm{End}_d(V) \underset{\text{def}}{\equiv} \{ \varphi : V \to V \\ \varphi : V_n \to V_{n+d} \\ n \in \mathbb{Z} \}.$$

$d \in \mathbb{Z}$

2) $J_n : V \longrightarrow \mathrm{End}_{-n}(V)$ linear map

~~$V_\Delta \longrightarrow \mathrm{End}_{-\Delta}(V)$~~

$J_0(|0\rangle) = id.$ $J_n(|0\rangle) = 0$ $n \neq 0$

$$J_n(v)|0\rangle = \begin{cases} 0 & n > -\Delta \\ v & n = -\Delta \end{cases}$$ $v \in V_\Delta$

3) $T(n) \equiv J_n(T)$ $n \in \mathbb{Z}$

$$[T(m), T(n)] = (m-n)\, T(m+n) + \tfrac{1}{12}(m^3 - m)\, \delta_{m+n,0}\, c_V\, id$$

$c_V \in \mathbb{C}$:

: commutation relations of Virasoro algebra.

4) $v \in V_\Delta$ $\quad J_n(T(-1)v)$

~~$[T(-1), J_n(v)]$~~ $= -(n+\Delta) J_n(v)$

~~$= -(n+\Delta-1) J_n(v)$~~.

5) Commutator relations

$v_1 \in V_{\Delta_1}$, $v_2 \in V_{\Delta_2}$

$$[J_m(v_1), J_n(v_2)] = \sum_{j=0}^{\Delta_1+\Delta_2-1} \binom{m+\Delta_1-1}{j} J_{m+n}(J_{j-\Delta_1+1}(v_1) v_2)$$

6) Associativity relations

$$J_m(J_n(v_1) v_2)$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n+\Delta_1-1}{j} \times$$

$$\{ J_{n-j}(v_1) J_{m-n+j}(v_2)$$

$$- (-1)^{n+\Delta_1-1} J_{m+\Delta_1-j-1}(v_2) J_{j-\Delta_1+1}(v_1) \}$$

II. Current Lie algebra $\mathfrak{g}(V)$

a) $\mathfrak{g}(V)^f = \sum_{d \in \mathbb{Z}} \mathfrak{g}_d(V)^f$

- graded Lie algebra
- generated by

  $J_n(v)$, $n \in \mathbb{Z}$, $v \in V$
- relation 4)
- commutator relations 5)

  $\deg J_n(v) = n$.

b) $\mathfrak{g}(V) = \varprojlim_n \mathfrak{g}(V)^f \big/ \sum_{d \geq n} \mathfrak{g}_d(V)^f$

- completion of $\mathfrak{g}(V)^f$
- $F^p(\mathfrak{g}(V)) = \varprojlim_n \sum_{d \geq p} \mathfrak{g}_d(V) \big/ \sum_{d \geq n} \mathfrak{g}_d(V)$.

  $\supseteq F^p \supseteq F^{p+1} \supseteq \cdots$

III current algebra $\mathcal{U}(V)$

a) $U(\mathfrak{g}(V))$

• completed Universal enveloping alg. of $\mathfrak{g}(V)$ by using filteration F^p.

b) ~~$U(\mathfrak{g}(V))/I$~~ $\mathcal{U}(V) = U(\mathfrak{g}(V))/I$

I: two sided ideal generated by associativity relation 6)

c) Filtration on $\mathcal{U}(V)$

$\cdots \supseteq F^p\mathcal{U}(V) \supseteq F^{p+1}\mathcal{U}(V) \supseteq \cdots$

: induced by $F^\cdot$ on $\mathfrak{g}(V)$.

$F^p\mathcal{U}(V) \cdot F^q\mathcal{U}(V)$

$\subseteq F^{p+q}\mathcal{U}(V)$

IV V- module

Def

M : V- module

$\Leftrightarrow$ M : . left $\mathcal{U}(V)$ module

. finitely generated

. $\forall m \in M$

$\dim F^0\mathcal{U}(V)\, m < \infty$

$\mathcal{M}_V$: category of V- modules

$\shortparallel$

$\mathcal{M}od(V)$

§2. Regularity conditions

I. Zhu's C_2 - condition

· Def. M: V- module. $n \geq 2$

$$O_n(M) = \{ J_{-\Delta-p}(v) M.\ v \in V_\Delta,\ p \geq n-1 \} \subseteq M$$

· Def. V- satisfies Zhu's C_2- condition

$\Longleftrightarrow \dim V / O_2(V) < \infty.$

II. Zero mod algebra $A_0(V)$.

Def. $A_0(V)$ (or Zhu-algebra)

$$A_0(V) \equiv F^0\mathcal{U}(V)/I^0$$

$I^0 \equiv I_1 \cap F^0\mathcal{U}(V)$:

I_1: ~~two sided~~ left ideal of $\mathcal{U}(V)$ generated by $F^1\mathcal{U}(V)$

Mod$(A_0(V))$: category of finite dim. $A_0(V)$ module.

Theorem

V satisfies C_2 condition

$\Longrightarrow \dim A_0(V) < \infty$.

V : satisfies condition I.

$$V = \mathbb{C}|0\rangle \oplus U \oplus O_2(V)$$

U: finite dim.

Theorem I. Fermionic prop.

$M \in \mathrm{Mod}(V)$. $M = \mathcal{U}(V)\cdot W$

W : finite dim. $F^0 U(V)$ modules

$\Longrightarrow$ M is linearly spanned by

$$\{ J_{-j_1}(v_1) \cdots J_{-j_N}(v_N)\, w \;.\; w \in W,\ v_1, \dots, v_N \in U \;.\; j_1 > \cdots > j_N > 0 \}$$

<u>Cor.</u> $M \in \mathrm{Mod}(V)$

$\Longrightarrow$ $\dim M/O_n(V) < \infty$ $\quad \forall\, \underline{n \geq 2}$

Cor. V satisfies Condition I.

- $M \in \mathfrak{M}_V$

$\Rightarrow \quad M = \sum_{\lambda \in P(M)} M_\lambda$

- generalized eigen space decomposition by $T(0)$

- $\dim M_\lambda < \infty$

- $P(M) \subseteq \{\lambda_1, \cdots \lambda_N\} + \mathbb{Z}_{\geq 0}$

$\lambda_1, \cdots \lambda_N \in \mathbb{C}$.

III. induced module.

- V satisfies Condition I.
- $W \in \mathcal{M}(A_0(V))$

Def

$$M(W) \equiv \mathcal{U}(V) \otimes_{F^0\mathcal{U}(V)} \overline{W}$$

$\overline{W}$: $F^0\mathcal{U}(V)$ module by $F^0\mathcal{U}(V) \to A_0(V)$

Prop.

$$M(W) \in \mathcal{M}(V)$$

VI Regularity condition

- Cond. (II) $A_0(V)$: semi-simple algebra

- Notation
  $\{W_i\}_{i \in I}$: the set of simple $A_0(V)$ module

- $|I| < \infty$ $\quad 0 \in I$

  $W_0 = \mathbb{C}|0\rangle$. $\begin{cases} J_0(|0\rangle)|0\rangle = |0\rangle \\ J_0(v)|0\rangle = 0 \quad v \in V_0 \end{cases}$

Condition III

$M_i \underset{\text{def}}{\equiv} M(W_i) \quad i \in I$

: irreducible V-module

$\therefore M_0 \cong V$.

V): Duality on $\mathcal{M}(V)$

a) $\theta : \mathfrak{g}(V)^{+} \longrightarrow \mathfrak{g}(V)^{+}$

$$J_n(v) \longrightarrow (-1)^{\Delta} J_n(e^{T(1)} v)$$

$v \in V_{\Delta}$.

Prop. Frenkel-Huang-Lepowsky

1) θ: anti-Lie alg. hom

2) $\theta^2 = id$

3) $M \in \mathcal{M}_V$ V satisfies condition I

$$D(M) = \sum_{\lambda} \mathrm{Hom}(M_\lambda, \mathbb{C})$$

$$\langle J_n(v)\varphi, u \rangle = \langle \varphi, \theta(J_n(v))u \rangle$$

$\Rightarrow$ $D(M) \in \mathcal{M}_V$

$D^2(M) = M$.

V satisfies condition I.

$\theta : A_0(V) \longrightarrow A_0(V)$

induced auto automorphism

$$D : M(A_0(V)) \longrightarrow M(A_0(V))$$
$$W \longrightarrow D(W) = Hom_{\mathbb{C}}(W)$$

Theorem II.

V : satisfied I. II. III.

$$\Longrightarrow M : \mathcal{M}(A_0(V)) \longrightarrow \mathcal{M}_V$$

equivalence of category.

§3. The space of Conformal Blocks

0) Notations $\mathbb{P}^1$-case.

- Assumption

V satisfies <u>I), II), III)</u>

- Situation
  - A : finite set $A \ni \infty$
  - $w_A = (w_a)_{a \in A} \subseteq \mathbb{P}^1 = \mathbb{C} \cup \infty$

  ($\cdot\ w_a \neq w_b$ if $a \neq b$
  ($\cdot\ w_\infty = \infty$

$$\begin{cases} \xi_a = z - w_a & a \neq \infty \\ \xi_\infty = [\text{scribbled out}]\ z, \end{cases}$$

local coordinate on $\mathbb{P}^1$ near w_a.

- $v \in V_\Delta$

$$J(v, \xi_a) = \sum_{n \in \mathbb{Z}} J_n(v) \xi_a^{-n-\Delta}$$

$$\hat{J}(v, \xi_\infty) = \sum_{n \in \mathbb{Z}} \theta(J_n(v)) \xi_\infty^{-n-\Delta}$$

- $f(z)(dz)^{1-\Delta} \in H^0(\mathbb{P}^1, \Omega^{1-\Delta}(*Q_\mathfrak{F}))$

  $f_a(\xi_a)(d\xi_a)^{1-\Delta}$

  : expansion of $f(z)(dz)^{1-\Delta}$ by ξ_a near w_a ⌋

II) Adel $\mathfrak{g}_A(V)$
Principal Adel $\mathfrak{g}^{out}_{\omega_A}$

<u>Def.</u>

- $\mathfrak{g}_A(V) \equiv \sum_{a \in A} \mathfrak{g}(V)_a$: $\mathfrak{g}(V)_a$ copy of $\mathfrak{g}(V)$

- $\mathfrak{g}^{out}_{\omega_A} \subset \mathfrak{g}_A(V)$: Lie sub alg.

$$\mathfrak{g}^{out}_{\omega_A} \equiv \{ J_A (v \otimes f(z)(dz)^{1-\Delta}) \}$$

$v \in V_\Delta$

$f(z)(dz)^{1-\Delta} \in H^0(\mathbb{P}', \Omega^{\otimes(1-\Delta)}(*A))$, f

$$J_A = \sum_a J_a$$

$$J_a(v \otimes f(z)\, dz^{1-\Delta}) = \mathrm{Res}_{\xi_a = 0} \left(J(v, \xi_a) f_a(\xi_a)\, d\xi_a \right).$$

$a = \infty$ $\quad J(v, \xi_a) \longrightarrow J^\theta(v, \xi_a)$

$\mathrm{Res}_{\xi_a = 0} \longrightarrow \mathrm{Res}_{\xi_a = \infty}$

II. Conformal Blocks

$\{M^a\}_{a \in A}$ $\quad M^a \in \mathcal{M}_V$

$M_A \equiv \bigotimes_a M^a$

$\mathfrak{g}_A^{\cdot}(V)$ module

Def.

$$\mathcal{V}_{w_A}(M_A) \equiv M_A \big/ \mathfrak{g}_{w_A}^{out}(M_A)$$

$$\mathcal{V}_{w_A}^{\dagger}(M_A) \equiv \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{w_A}(M_A), \mathbb{C})$$

Theorem III. Finite dimensionality

$\dim_{\mathbb{C}} V^{+}_{w_A}(M_A) < \infty$.

Problems

1). How $V^{+}_{w_A}(M_A)$ moves when w_A moves

- $\dim V^{+}_{w_A}(M_A)$ independent to w_A?

2) How to compute $\dim_{w_A} V^{+}_{w_A}(M_A)$.

§4 Main results.

I) compactification of Moduli-space.

$A \ni \{0, 1, \infty\}$

$X_A = \{ w_A = (w_a) \in (P^1)^A$

$w_a \neq w_b \quad a \neq b$

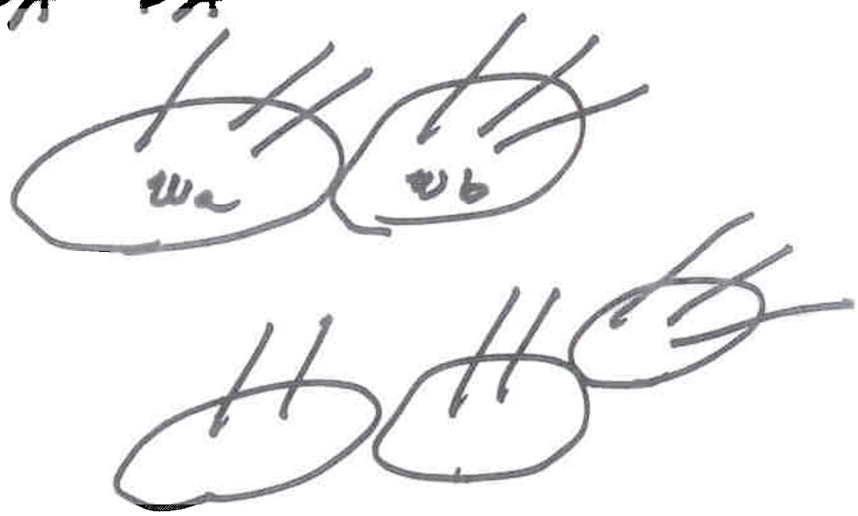
$w_0 = 0, w_1 = 1, w_\infty = \infty \}$

$\overline{X}_A = X_A \cup D_A$. compactification

$\overline{X}_A \supseteq D_A$: normal crossing divisor

$D_A \ni w_A$: represent degenerate points

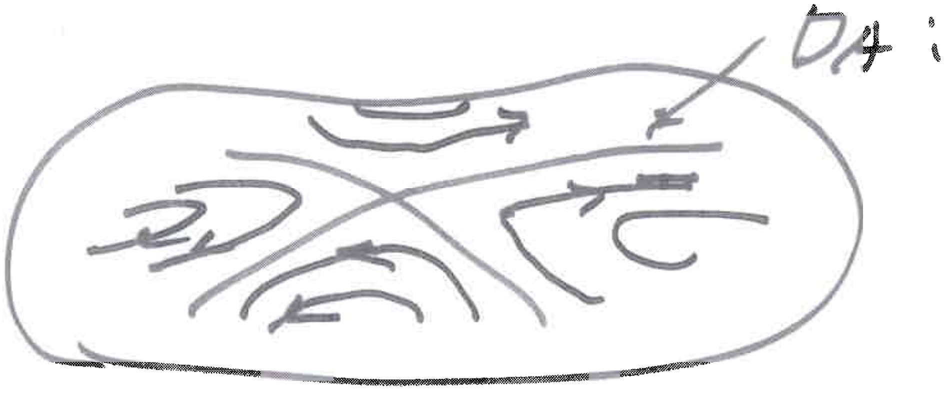
$w_A \in D_A$



etc.

$\Theta_{\overline{X}_A}(-\log D_A)$: the sheaf of holomorphic vector field on $\overline{X}_A$ tangent to D_A

$\overline{X}_A$



II) The sheaf

Def. $\mathcal{V}_{\bar{X}_A}(M_A) = \bigcup_{\omega_A \in \bar{X}_A} \mathcal{V}_{\omega_A}(M_A)$

Theorem IV Gauss-Manin connection

a) $\mathcal{V}_{\bar{X}_A}(M_A)$: coherent $\mathcal{O}_{\bar{X}_A}$ module

b) $\nabla : \Theta_{\bar{X}_A}(-\log D_A) \otimes \mathcal{V}_{\bar{X}_A}(M_A)$

$\longrightarrow \mathcal{V}_{\bar{X}_A}(M_A)$

integrable connection

$\nabla_{\partial/\partial z_a}([u])$

$= [P_a(T(\partial))u]$, $u \in M_A$.

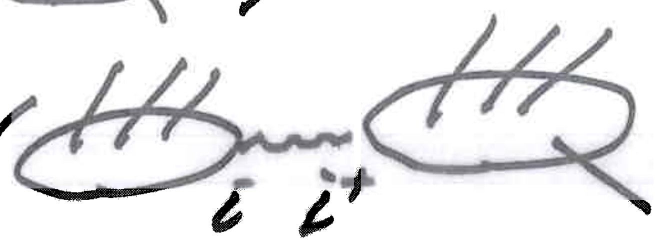
Theorem V. locally freeness.

$\mathcal{V}_{X_A}(\mathcal{H}_A)$

locally free $\mathcal{O}_{X_A}$ modules

Theorem VI Factorization

$\mathcal{V}($  $)$

$\cong \sum_{i \in I} \mathcal{V}($  $)$

$\boxed{M_{i^+} = D M_i}$

Theorem VII

a) $\mathcal{M}_V$: semi-simple rigid, braided. tensor category

b) $\mathcal{M}_V \cong \mathcal{M}(A_0(V))$

$A_0(V)$: "quantum group"

§5 Further Problem

I. Extend to general Rieman surfaces

O.K.

II. What is the relations between conditions I. II. III.

III. Construct Further example

∘ minimal series for W-algebras $W(\mathfrak{g})$ associated to simple Lie alg. $\mathfrak{g}$.

The end