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MSRI.

Gauss-Selberg Sums

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I. Gauss-Selberg integrals

(Euler formula)

$$\int_{-\infty}^{\infty} e^{-x^2} |x|^{2k} dx = \Gamma(k+\tfrac{1}{2}) \qquad (\mathrm{Re}\, k \geq -\tfrac{1}{2})$$

(Generalization)

$$\int_{\mathbb{R}^n} e^{-|x|^2/2} \prod_{\alpha\in R} |x_\alpha|^k dx = \prod_{\alpha\in R_+} \frac{(k(\rho,\alpha^\vee)+k)!}{(k(\rho,\alpha^\vee))!}$$

$R \subset \mathbb{R}^n$ reduced irreducible $(\ ,\ )$.

$$x_\alpha = \frac{\sqrt{2}(x,\alpha)}{|\alpha|}, \quad \rho = \frac{1}{2}\sum_{\alpha>0} \alpha, \quad \mathrm{Re}\, k \geq -\frac{1}{h}$$

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$

— Selberg – Mehta – Macdonald – Bombieri – Opdam

Definition let F — "perfect" Fourier transform with γ = Gaussian

$$F(\gamma) = g\,\gamma^{-1}$$

$g \overset{def}{=}$ Gauss-Selberg sum.

traditional normalization : $F(\text{delta function}) = 1$.

Hankel transform

Def. $\varphi(x, \lambda)$ — even w.r.t. x. $\varphi(0, \lambda) = 1$.

$$\left(\frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx} \right) \varphi_\lambda(x) = 4\lambda^2 \varphi_\lambda(x)$$

$$\varphi_\lambda(x) = \tilde{\varphi}(\lambda x)$$

$\tilde{\varphi}$: Bessel function. $\left(\tilde{\varphi}'' + \frac{2k}{x}\tilde{\varphi}' = 4\tilde{\varphi} \right)$.

$$H(f(x)) = 2\int_0^\infty \varphi_\lambda(x) f(x)\, x^{2k}\, dx$$

Theorem. $2\int_0^\infty \varphi_\lambda(x)\varphi_\mu(x)\, e^{-x^2} x^{2k}\, dx$

$$= \varphi_\lambda(\mu)\, e^{\lambda^2+\mu^2}\, \Gamma(k+\tfrac{1}{2}).$$

$$\mu = 0 \Rightarrow H(\gamma) = \gamma^{-1}\Gamma(k+\tfrac{1}{2}), \quad (\gamma = e^{-x^2})$$

Trigonometric theory

Theorem 1. $\boxed{0<q<1, \quad q=e^{-1/a}, \quad a>0}$

$$\boxed{\frac{1}{i}\int_{i\mathbb{R}} q^{-x^2}\,\delta_k(x)\,dx = \sqrt{2\pi a}\prod_{j=0}^{\infty}\frac{1-q^{k+j}}{1-q^{2k+j}}} \qquad \left(\approx \frac{\Gamma_q(2k)}{\Gamma_q(k)}\right),$$

where $\delta_k(x) = \prod_{j=0}^{\infty} \frac{1-q^{j+2x}}{1-q^{j+2x+k}}\,\frac{1-q^{j-2x}}{1-q^{j-2x+k}}$

$$\delta_k(x) \underset{q\to 0}{\rightsquigarrow} x^{2k} \quad (\text{Stirling-Moak})$$

II GAUSS SUMS

(Gauss). $q = \exp\left(\frac{2\pi i}{N}\right)$.

$$\sum_{m=0}^{2N-1} q^{\frac{m^2}{4}} = (1+i)\sqrt{N}$$

Gauss-Selberg sum $\overset{\shortparallel}{g} = \frac{F(\gamma)}{\gamma^{-1}}$ $\quad (F(\gamma) = g\,\gamma^{-1})$

F – Fourier transform on $\mathrm{Funct}(\mathbb{Z}_{2N})$

$$\hat{f}(\lambda) = \sum_{x\in\mathbb{Z}_{2N}} q^{\lambda x/2} f(x), \quad \gamma = q^{x^2}$$

Theorem 2

$$\sum_{j=0}^{\infty} q^{\frac{(j-k)^2}{4}} \left(\frac{1-q^{k+j}}{1-q^k}\right)\left(\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l}\right).$$

follows from ${}_6\psi_6$ $\longrightarrow$ $= \prod_{j=1}^{\infty}\frac{1-q^{k+j}}{1-q^j}\sum_{j=-\infty}^{\infty} q^{\frac{(j-k)^2}{4}}$.

The L.H.S. as $k\in\mathbb{R}_+$ is

$$\left(= \int q^{+x^2}\,\delta_k(x)\,dx \right.$$

up to normalization

<u>Theorem 3</u>. Assume $k \in \mathbb{Z}$, $q^{1/4}$ - primitive N^{th} root of unity.

Above expression becomes, after classical procedure,

$$\sum_{j=0}^{N-2k} q^{\frac{(j-k)^2}{4}} \left(\frac{1-q^{k+j}}{1-q^{k}} \right) \left(\prod_{l=1}^{j} \frac{1-q^{l+2k-1}}{1-q^{l}} \right)$$

$$= \prod_{j=1}^{k} \frac{1}{1-q^j} \underbrace{\sum_{m=0}^{2N-1} q^{\frac{m^2}{4}}}_{\overset{\text{let}}{=}\, g}$$

g is the generalized GAUSS sum.

<u>Cor</u>. $N = 2n$

$k \overset{\text{let}}{=} n = N/2$

Then it follows from above that

$$q^{n^2/4} = \prod_{j=1}^{n} \frac{1}{1-q^j} \sum_{j=0}^{2N-1} q^{j^2/4}$$

$$\Rightarrow \boxed{ g \overset{N=2n}{=} q^{\frac{n^2}{4}} \prod_{j=1}^{n} (1-q^j) . }$$

very $\Downarrow$ simple

$g = (1+i)\sqrt{N}$ as $q = \exp(2\pi i/N)$

We change $q^{1/4} \longrightarrow -q^{1/4}$ in Theorem 3 and get

$\boxed{N=2n+1}$ $\quad 0<k<n \ , \quad k\in\mathbb{Z}$

$$\sum_{j=0}^{n-k} \frac{1-q^{2j+k}}{1-q^k} \prod_{\ell=1}^{2j} \frac{1-q^{\ell+2k-1}}{1-q^\ell}\, q^{j^2-kj} = \prod_{j=1}^{n-k} \frac{1-q^{2k+2j}}{1+q^{k+2j}} \quad , \ q \text{ - primitive } N\text{-th root}$$

Let $q^{1/2}=q^{-n}$ (i.e. $q^{n+1/2}=1$)

$k \longmapsto \boxed{\bar{k} \overset{\text{def}}{=} k-n-1/2}$ $\quad (\bar{1}=-n+\frac{1}{2})$

$\boxed{\delta = n-k} \Rightarrow \boxed{\bar{k}=-1/2-\delta}$

$$\sum_{j=0}^{\delta} \frac{1-q^{2j+\bar{k}}}{1-q^{\bar{k}}} \prod_{\ell=1}^{2j} \frac{1-q^{\ell+2\bar{k}-1}}{1-q^\ell}\, q^{j^2-\bar{k}j} \overset{!}{=} \prod_{j=1}^{\delta} \frac{1-q^{2\bar{k}+2j}}{1+q^{\bar{k}+2j}}$$

$\boxed{\text{FOR ALL } q\in\mathbb{C}}$ (not roots of unity)

III DAHA & Verlinde algebras

Definition. $\mathcal{H} = \langle X^{\pm 1}, Y^{\pm 1}, T \rangle$

$$\begin{cases} TXT = X^{-1} \\ T^{-1}YT = Y^{-1} \\ Y^{-1}X^{-1}YXT^{2} = q^{-1/2} \end{cases} \qquad (T - t^{1/2})(T + t^{-1/2}) = 0, \quad t = q^{k}$$

degenerate case:

$\underline{t = 1}$ $\qquad T \to s$

$$\Rightarrow \quad \mathcal{W} := \begin{cases} s^2 = 1 \\ sXs = X^{-1} \\ sYs = Y^{-1} \\ Y^{-1}X^{-1}YX = q^{-1/2} \end{cases}$$

$\mathcal{W}$ = Weyl algebra extended with s.
It has a unique irrep V_{2N} as q acts by the multiplication by q.

$$V_{2N-4k} = \mathrm{Funct}\left(\frac{-N+k+1}{2}, \ldots, \frac{-k}{2}, \frac{k+1}{2}, \ldots, \frac{N-k}{2}\right)$$

$0 < k < N/2$ step $= 1/2$

$$Tf(z) = \frac{t^{1/2}q^{2z} - t^{-1/2}}{q^{2z}-1} f(s(z)) - \frac{t^{1/2}-t^{-1/2}}{q^{2z}-1} f(z)$$

$Xf(z) = q^z f(z)$, $pf(z) = f(z+1/2)$, $s(z) = -z$

$Yf(z) = spT f(z)$ IS IRREP OF $\mathcal{H}$

VERLINDE $=$ V^{even}_{2N-4k} $\quad \dim = N-2k+1$

$N =$ level $+2$ $\quad k=1 \Rightarrow \dim = N-1$

$$\boxed{N = 2n+1}$$

$$V'_{N-2k} = \mathrm{Funct}\left(\frac{-N+k+1}{2}, \ldots, \frac{-k}{2}, \frac{k}{2}+1, \frac{N-k-1}{2}\right)$$

step $= 1$

module over X^2, Y^2, T HAS q-DEFORMATIONS

SMALL VERLINDE : $V'^{\,\text{even}}_{N-2k}$ $\quad \dim = n-k+1$

(root lattice) $\quad k=1 \Rightarrow \dim = n$

The S-operator: acts on

$$S = \tau_+ \tau_-^{-1} \tau_+ : \mathcal{H}, V_{2N-4k}, V^{\text{even}}_{2N-4k}$$

where $\tau_+ : \begin{matrix} X \to X \\ Y \to XYq^{-1/4} \end{matrix}$, $\tau_- : \begin{matrix} Y \to Y \\ X \to YXq^{1/4} \end{matrix}$ $\quad T \to T$

(the GAUSSIAN) $\quad (q, t \to q, t)$

V^{even}_{2N-4k} is a generalized Verlinde alg,
with $\dim = N-2k+1$ (generalization : adding k)
$k=1 \Rightarrow$ Classical Verlinde
(Kac-Peterson-Cherednik-Verlinde),
$k>1$ (Kirrilov Jr, Cherednik)

IV. Hankel, osp(2,1)

Let $N=2n+1$, $\bar{k} = k-n-1/2$, $q^{1/2} = q^{-n}$.

The deformation of V'_{N-2k} is an $\mathcal{H}(\bar{k})$ module $\tilde{V}'_{2\delta+1}$ for $k=-\frac{1}{2}-\delta$
of dimension $2\delta+1$ $(=N-2k)$

Let $q = e^{-\hbar}$, $\hbar \to 0$,

$X = e^{+\hbar x}$ $\quad t = q^k$

$Y = e^{+\hbar y}$ $\quad T \to s$ $(T^2 \approx 1 - \hbar k s)$

$\boxed{\text{Theorem 4 } S\text{-operator in the limit becomes Hankel transform and the Weyl element in } SL_2(\mathbb{C})}$

$\to$ rational degeneration

$$\mathcal{H}''_k \stackrel{\text{def}}{=} \langle x, y, s\rangle, \text{ with } \begin{cases} sxs = -x \\ sys = -y \\ [y,x] = 1+2ks \end{cases}$$

This alg is directly connected to osp(2,1)

$$\boxed{\begin{aligned} h &\stackrel{\text{def}}{=} xy+yx \\ [h,x] &= x \\ [h,y] &= -y \end{aligned}}$$

$\tilde{V}'_{2\delta+1}$ becomes $(2\delta+1)$-dim irrep for osp(2,1)

$\tilde{V}'^{\,\text{even}}_{2\delta+1}$ becomes $(\delta+1)$-dim irrep of sl_2

$$\left.\begin{cases} e = x^2 \\ h \\ f = -y^2 \end{cases}\right\} \to sl_2 \text{ relations}$$

f becomes $-\frac{1}{4}\left(\frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx}\right)$,

The S-operator becomes the Hankel transform for "singular" Bessel functions as $k=-1/2-\delta$. } Weyl element of SL_2