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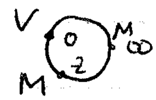
Tensor categories and orbifold vertex algebras

V — a vertex operator algebra $\left\{ \begin{array}{l} V \text{ inf dim vec. space} \\ Y_z : V \otimes V \to \overline{V} \leftarrow \text{completion w.r.t grading} \end{array} \right.$

single valued func of z

M_V — modules over V

$\left\{ Y_z : V \otimes M \to \overline{M} \right.$



single valued function of z.

G — finite group action on V

invariants $V^G \subset V$ is also a vertex operator algebra

<u>Problem</u> : Given V and G,

- $M_{V^G} = ?$

- Special case : $M_V = Vec$ (V has unique simple module, V itself).
("holomorphic case")

<u>Want</u> M_V to be "good,"
namely, $\otimes$, semisimple, $*$ (dual) (rigidity)
modular.

<u>From now</u>, Assume our VA is good enough so that M_V has these properties.

• DVVV '89 (Dijkgraaf, Vafa, Verlinde, Verlinde)

In holom. case, $\mathcal{M}_{V^G} = D^{\omega}(G)$-mod

$\omega = 1$: D^{ω}-mod = Vector spaces with two structures

$M = \bigoplus M_g$ graded

action of G s.t. $h \cdot M_g \subset M_{hgh^{-1}}$

Obvious : ① No action of G on $M \in \mathcal{M}_V$

But given M, can form a new module

$M \longrightarrow M^g$

(= M as vector space but action of V twist by g)

② We have restriction functor.

$\mathrm{Res} : \mathcal{M}_V \to \mathcal{M}_{V^G}$

• not a $\otimes$ functor (Consider unit V, V^G respectively)

• not every $X \in \mathcal{M}_{V^G}$ appears as a summand in $\mathrm{Res}(M)$, $M \in \mathcal{M}_V$.

"untwisted sector": those $X \subset \mathrm{Res}(M)$, $M \in \mathcal{M}_V$

Denote: $\mathcal{M}^1_{V^G}$

* **Assume**: M_{V^G} is modular. (non holom. case)

Theorem 1 $\quad M'_{V^G} = M_V /\!/ G = \left\{ M \in M_V,\ \{\varphi_g : M \xrightarrow{\sim} M^g\}_{g \in G} \right\}$

$\varphi_{g_1}\varphi_{g_2} = \varphi_{g_1 g_2}$

(as linear maps on vector space M)

Note: if $M^g \neq M$, $g \neq 1$ for simple M, then

$$(\text{simple obj. in } M_V/\!/G) = (\text{simple obj in } M_V)/G$$

Ex. If $M_V = \mathrm{Vec}$,

$\mathrm{Vec}/\!/G = \mathrm{Rep}\, G \qquad (\text{analog of } pt/\!/_G = BG)$

Theorem 2 $\quad M_{V^G} = \left(\bigoplus_{g \in G} M_V^g \right) /\!/ G$

where M_V^g = "g-twisted modules" over V

$$\begin{array}{ccc} Y_z : V \otimes M & \longrightarrow & M \\ \downarrow & &)) \\ V \otimes M & \xrightarrow{g \otimes 1} & M \end{array}$$

if we go around the origin, monodromy.

Theorem 3 If $M_V = \mathrm{Vec}$, then

- $\bigoplus_{g \in G} M_V^g \simeq G$-graded vector spaces (up to cocycle)
- $\left(\bigoplus M_V^g\right) /\!/ G = D^\omega(G)$-mod.

Ideas of proof

Denote $\mathcal{C} = M_{V^G}$

Consider V as a module $/ V^G$. Write $A = V$
$A \in \mathcal{C}$

Then: $Y_z : V \otimes V \to \bar{V}$ gives rise to map

$A \underset{\mathcal{C}}{\otimes} A \to A$, which is comm. assoc, and has unit.

We also have:

- $A^G = \mathbb{1}$ $\qquad \mathbb{1} = V^G$ – the unit object in M_{V^G}
- $[A : \mathbb{1}] = 1$
- product is "non-degenerate".

Lemma 1. For $\mathcal{C}$ – BTC (braided tensor category) rigid,
A – an algebra in $\mathcal{C}$ satisfying above properties

Then $\mathcal{C} \supset \mathrm{Rep}\, G$ $\qquad$. $\varphi : \mathrm{Rep}\, G \to \mathcal{C}$ $\qquad$ follows from duality
$\qquad \cup$
$A = \dots \qquad \mapsto (\ \otimes A)^G$.

<u>Lemma 2</u> : One has natural functors

$$F : \mathcal{C} \longrightarrow A\text{-mod} \qquad \text{"induction"}$$
$$M \mapsto A \underset{\mathcal{C}}{\otimes} M$$
$$G : A\text{-mod} \longrightarrow \mathcal{C} \qquad \text{"restriction"}$$

F is $\otimes$ functor ; G is not.

Def $M \in A$-mod is
$M \in \mathcal{C}$
$A \underset{\mathcal{C}}{\otimes} M \to M$
w/ compatibility conditions

Suppose $M \in A$-mod.
It is called g-twisted if

$$\begin{tikzcd} A \otimes M \arrow[r] \arrow[d, "R^2"] & M \\ A \otimes M \arrow[r, "g \otimes 1"] & A \otimes M \arrow[u] \end{tikzcd}$$

commutative

<u>Lemma 3</u> : $A\text{-mod} = \bigoplus_g (A\text{-mod})_{g\text{-twisted}}$.

Theorems follow from these lemmas