Representation theory beyond affine Lie algebras: A perspective

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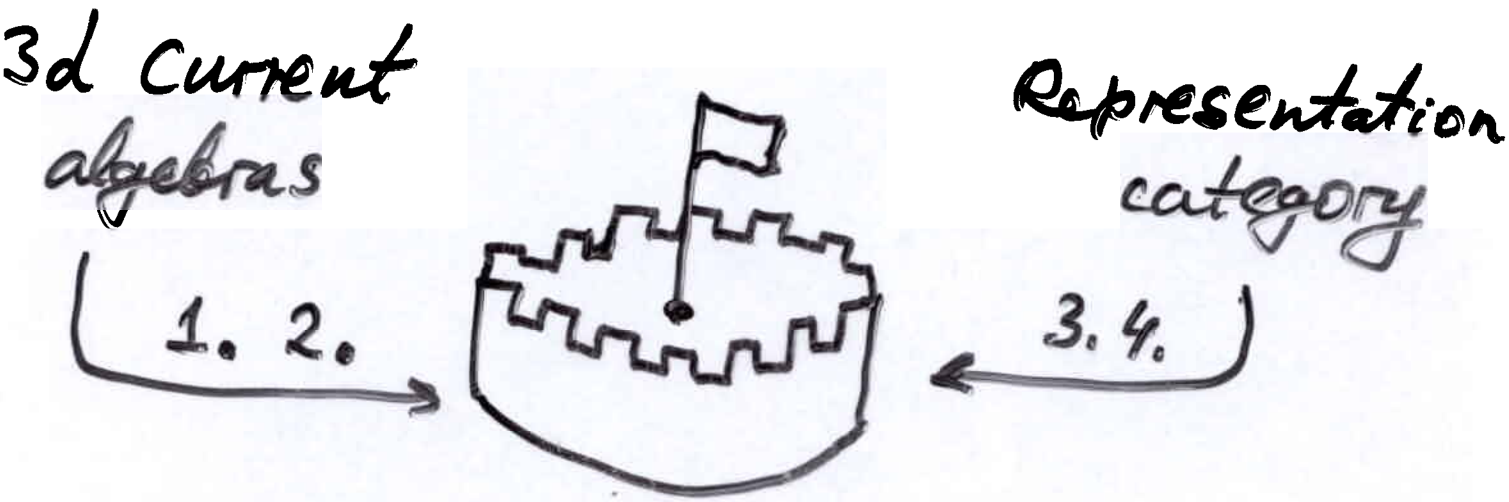
$$\begin{array}{ccc} \mathrm{Map}(S^1, \mathfrak{g})^\wedge & \cong & \text{Kac-Moody}\,(\hat{\mathbb{C}}) \\ \downarrow & & \downarrow \\ \boxed{\mathrm{Map}(X^3, \mathfrak{g})^\wedge} & & \text{Kac-Moody}\,(GC) \end{array}$$



1. Representations of convolution algebras

$$\mathfrak{gl}(\infty)^\wedge \cong Map(S^1 \times S^1, \mathbb{C})^\wedge$$

$$E_{mn} \longleftrightarrow e^{ims_1 - ims_2}$$

$s_1, s_2 \in \mathbb{R}/2\pi\mathbb{Z}$

Spinor representation : Sato et al.

We study spinor representation of

$$Map(S^3 \times S^3, \mathbb{H})^\wedge$$

$$(F \circ G)(s_1, s_2) = \frac{1}{2\pi^2} \int_{S^3} F(s_1, s) G(s, s_2) d^3 s$$

Central extension arises in representation

$$\mathrm{Map}(S^3 \times S^3, \mathbb{H}) \cong \mathrm{Map}_{reg}(\mathbb{H}^\times \times \mathbb{H}^\times, \mathbb{H})$$

$$\vec{D}_z F(z,w) = 0 = F(z,w)\overleftarrow{D}_w$$

$$\vec{D}_z F \overset{\mathrm{def}}{=} \frac{\partial F}{\partial z_0} + i\frac{\partial F}{\partial z_1} + j\frac{\partial F}{\partial z_2} + k\frac{\partial F}{\partial z_3}, \quad z = z_0 + iz_1 + jz_2 + kz_3$$

$$F\overleftarrow{D}_w \overset{\mathrm{def}}{=} \frac{\partial F}{\partial w_0} + \frac{\partial F}{\partial w_1}i + \frac{\partial F}{\partial w_2}j + \frac{\partial F}{\partial w_3}k, \quad w = w_0 + iw_1 + jw_2 + kw_3$$

$$(F \circ G)(z', z'') = \frac{1}{2\pi^2}\int_{S^3} F(z', z)\, d^3z\, G(z, z'')$$

$$d^3z = dz_1 \wedge dz_2 \wedge dz_3 + i\, dz_0 \wedge dz_1 \wedge dz_2 + j\, dz_0 \wedge dz_1 \wedge dz_3 + k\, dz_0 \wedge dz_1 \wedge dz_2$$

cf.

$$\mathrm{Map}(S^1 \times S^1, \mathbb{C}) \cong \mathrm{Map}_{reg}(\mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C})$$

$$\bar{\partial}_z F(z,w) = 0 = \bar{\partial}_w F(z,w)$$

$$(F \circ G)(z', z'') = \frac{1}{2\pi i}\int_{S^1} F(z', z) G(z, z'')\, dz$$

$\mathbb{H} \cong \mathbb{C}^2$ spinors, denoted S

$S \otimes S^* \cong \mathbb{H}_{\mathbb{C}} \qquad (s \otimes s^*)(s') = s \langle s^*, s' \rangle$

$V \stackrel{\text{def}}{=} \mathrm{Map}_{reg}(\mathbb{H}^\times, S)$, i.e. $\vec{D} f = 0$

$V^* \stackrel{\text{def}}{=} \mathrm{Map}_{reg}(\mathbb{H}^\times, S^*)$, i.e. $g \overleftarrow{D} = 0$

$V = \bigoplus_{n \in \mathbb{Z}} V(n) \quad , \quad V^* = \bigoplus_{n \in \mathbb{Z}} V^*(n)$

graded by degree operator $d = \sum_{i=0}^{3} z_i \frac{\partial}{\partial z_i}$.

$\langle V^*(m), V(n) \rangle$ perfect pairing $m+n+3=0$

$\langle g, f \rangle \stackrel{\text{def}}{=} \frac{1}{2\pi^2} \int_{S^3} g(z) d^3z \, f(z)$

cf. $V = \mathrm{Map}_{reg}(\mathbb{C}^\times, \mathbb{C})$, $V^* = \mathrm{Map}_{reg}(\mathbb{C}^\times, \mathbb{C})$, $z\frac{d}{dz}$

$\langle V^*(m), V(n) \rangle$ pairing $m+n+1=0$, $\frac{1}{2\pi i} \int_{S^1} g(z) f(z) dz$

$\{f_i(z)\}_{i\in I}$ basis of V, $\{g_i(z)\}_{i\in I}$ dual basis of V^*

$Cl(V\oplus V^*)$ Clifford algebra, $\{\beta_i\}_{i\in I}$, $\{\gamma_i\}_{i\in I}$ generators

$$\{\beta_i,\gamma_j\}=\delta_{ij}, \quad \{\beta_i,\beta_j\}=0, \quad \{\gamma_i,\gamma_j\}=0, \quad i,j\in I$$

$$\beta(z)\stackrel{\text{def}}{=}\sum_{i\in I}\beta_i g_i(z) \quad , \quad \gamma(z)\stackrel{\text{def}}{=}\sum_{i\in I}\gamma_i f_i(z)$$

$H=\Lambda(V_-\oplus V_+^*)$ Clifford module

$$\beta_i 1=0 \,, \quad i\in I_+ \; ; \quad \gamma_i 1=0 \,, \quad i\in I_-$$

<u>Theorem:</u> Spinor representation of $\mathrm{Map}(S^3\times S^3, \mathbb{H})^\wedge$ is given by

$$\pi(F)=\frac{1}{(2\pi^2)^2}\int_{S^3}\int_{S^3} :\beta(z)\,d^3z\,F(z,w)\,d^3w\,\gamma(w):$$

Proof is based on Cauchy-Fueter formulas

$$f(w) = \frac{1}{2\pi^2} \int_{S^3} k(z-w)\, d^3z\, f(z), \qquad \vec{D} f = 0$$

$$g(w) = \frac{1}{2\pi^2} \int_{S^3} g(z)\, d^3z\, k(z-w), \qquad g \overleftarrow{D} = 0$$

where $k(z-w) = \dfrac{(z-w)^{-1}}{|z-w|^2}$ comes from

$$\gamma(z)\beta(w) = :\gamma(z)\beta(w): + \; k(z-w), \qquad |z| > |w|$$

cf. the use of the Cauchy formula

$$f(w) = \frac{1}{2\pi i} \int_{S^1} \frac{f(z)}{z-w}\, dz$$ and norm. ord.

$$\gamma(z)\beta(w) = :\gamma(z)\beta(w): + \frac{1}{z-w}, \qquad |z| > |w|$$

in spinor representation of $Map(S^1 \times S^1, \mathbb{C})$

We can obtain a representation of

- $\mathrm{Map}(S^1, \mathbb{C})^\wedge$ Heisenberg algebra

$$\pi(F) = \frac{1}{2\pi i}\int_{S^1} :\beta(z)F(z)\gamma(z): dz$$

- $\mathrm{Vect}(S^1)^\wedge$ Virasoro algebra

$$\pi(l_n) = \frac{1}{2\pi i}\int_{S^1} :\beta(z)\left(z^{n+1}\frac{d}{dz} + \frac{n+1}{2}z^n\right)\gamma(z): dz =$$

$$= -\frac{1}{2\pi i}\int_{S^1} :\left(z^{n+1}\frac{d}{dz} + \frac{n+1}{2}z^n\right)\beta(z)\,\gamma(z): dz$$

In 3dimensional case only

- $$\pi(1) = \frac{1}{2\pi^2}\int_{S^3} :\beta(z)\, d^3z\, \gamma(z):$$

- $$\pi(X) = \frac{1}{2\pi^2}\int_{S^3} :\beta(z)\, d^3z\, (X\gamma)(z):$$

$$= -\frac{1}{2\pi^2}\int_{S^3} :(X\beta)(z)\, d^3z\, \gamma(z):$$

$X \in o(5,1)$

are well-defined

Generalizations

- $\mathrm{Map}(X \times X, \mathbb{H})^{\wedge}$, $X = S^3/\Gamma$, $\Gamma \subset SU(2)$
- $\mathrm{Map}(S^3 \times S^3, \mathrm{Mat}_{n\times n}(\mathbb{H}))^{\wedge}$
- $\mathrm{Map}(S^3 \times S^3, \mathbb{H})^{\wedge}_A$, $D_A = D + A$

<u>Main problem</u>: Construct spinor representation of $\mathrm{Map}(S^3, \mathbb{H})^{\wedge}$ and its generalizations

<u>Main obstacle</u>: Divergence of the central extension $\sim \frac{1}{\varepsilon^2}$, where ε regularization parameter

2. Deformation of classical equations in 4d.

There is a strong evidence that all the constructions of §1 admit a quantum deformation based on $SU(2)_q$

The direction of deformation depends on a choice of the identification

$$\mathbb{H} \overset{S^2}{\cong} \mathbb{C}^2$$

Conjecture: Representations of 3d current algebra can be constructed in a bundle of Clifford modules over the space of deformations S^2

Deformation of quaternions.

Recall quantum groups $GL(2)_q$, $SL(2)_q$, $U(2)_q$, $SU(2)_q$

Generators: $g_{11}, g_{12}, g_{21}, g_{22}$

Relations: $GL(2)_q$

$$g_{11}g_{12}=q^{-1}g_{12}g_{11} \qquad g_{11}g_{21}=q^{-1}g_{21}g_{11}$$

$$g_{12}g_{22}=q^{-1}g_{22}g_{12} \qquad g_{21}g_{22}=q^{-1}g_{22}g_{11}$$

$$g_{12}g_{21}=g_{21}g_{12}$$

$$g_{11}g_{22}-q^{-1}g_{12}g_{21}=g_{22}g_{11}-qg_{21}g_{12}\left(\stackrel{\text{def}}{=}\det_q\right)$$

$$SL(2)_q=GL(2)_q/\langle \det_q=1\rangle$$

Comultiplication:

$$\Delta T=T\otimes T \qquad T=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$$

etc.

Extensions $\widetilde{GL(2)}_q$, $\widetilde{SL(2)}_q$ of $GL(2)_q$, $SL(2)_q$

by adjoining an additional generator δ

$$\delta g_{ij} = q^{j-i} g_{ij} \delta \qquad i,j = 1,2$$

$$\Delta \delta = \delta \otimes \delta$$

Real structure : antilinear involution

$$g_{11}^* = g_{22} \quad g_{12}^* = -g_{21} \qquad \delta^* = \delta$$

$$g_{21}^* = -g_{12} \quad g_{22}^* = g_{11} \qquad (\delta^{-1})^* = \delta^{-1}$$

$$U(2)_q = \langle GL(2)_q , * \rangle, \quad SU(2)_q = \langle SL(2)_q , * \rangle$$

$$\widetilde{U(2)}_q = \langle \widetilde{GL(2)}_q , * \rangle, \quad \widetilde{SU(2)}_q = \langle \widetilde{SL(2)}_q , * \rangle$$

Two subalgebras of $\widetilde{SU}(2)_q$

$\mathbb{H}_q \equiv \mathbb{H}_q(0)$, and $\mathbb{H}_q(\infty)$

cf. nondeformed case $\mathbb{H}(0)$ and $\mathbb{H}(\infty)$

$\mathbb{HP}' = \mathbb{H}(0) \cup \mathbb{H}(\infty)$, $\mathbb{H}^\times = \mathbb{H}(0) \cap \mathbb{H}(\infty)$

Generators : $x_{ij} = q^{\frac{i-j}{2}} \delta g_{ij} = q^{\frac{j-i}{2}} g_{ij} \delta$

$y_{ij} = q^{\frac{j-i}{2}} \delta^{-1} g_{ij} = q^{\frac{i-j}{2}} g_{ij} \delta^{-1}$

Relations :

$x_{11} x_{12} = x_{12} x_{11}$, $x_{21} x_{22} = x_{22} x_{21}$,

$[x_{11}, x_{22}] + [x_{21}, x_{12}] = 0$, + 3 relations depending on q

$y_{11} y_{21} = y_{21} y_{11}$, $y_{12} y_{22} = y_{22} y_{12}$

$[y_{11}, y_{22}] + [y_{12}, y_{21}] = 0$, + 3 relations depending on q.

Algebra of forms on $\mathbb{H}_q(0)$, $\mathbb{H}_q(\infty)$

Faddeev-Reshetikhin-Takhtajan presentation

$R\,T_1 T_2 = T_2 T_1 R$ relations for $GL(2)_q$

$dT \stackrel{\text{def}}{=} \begin{pmatrix} dg_{11} & dg_{12} \\ dg_{21} & dg_{22} \end{pmatrix}$ matrix of 1-forms

Relations for $\Lambda^* GL(2)_q$

Wess-Zumino
Woronowicz
Manin, Sudbery

$R\,T_1\,dT_2 = dT_2\,T_1\,(R^t)^{-1}$

$R\,dT_1\,dT_2 = -dT_2\,dT_1\,(R^t)^{-1}$

Cauchy-Fueter integral formula has a quantum counterpart that uses noncommutative 3d residue (F-Takhtajan, in progress)

Deformation of ASDYM equations

joint work with M. Jardim

Main result: Quantum instantons have classical moduli spaces

Recall: Atiyah-Drinfeld-Hitchin-Manin data

V, W Hermitian vector spaces

$B_1, B_2 \in \operatorname{End} V$, $i \in \operatorname{Hom}(W, V)$, $j \in \operatorname{Hom}(V, W)$

satisfy the relations

$$[B_1, B_2] + ij = 0$$

$$[B_1, B_1^*] + [B_2, B_2^*] + ii^* - j^* j = 0$$

$$g \cdot (B_1, B_2, i, j) = (g B_1 g^{-1}, g B_2 g^{-1}, g i, j g^{-1})$$

$g \in U(V)$

$\mathcal{M}(n,k) = \{(B_1 B_2\, i\, j) \mid \text{relations}\} / U(V)$

$k = \dim V$, $n = \dim W$, ideal instantons

$\mathcal{M}^{reg}(n,k) \subset \mathcal{M}(n,k)$ moduli of instantons

$(B_1 B_2\, i\, j)$ is regular if stabilizer is triv.

Construction of quantum instantons

$$V \otimes \mathbb{H}_q \xrightarrow{\alpha} \tilde{W} \otimes \mathbb{H}_q \xrightarrow{\beta} V \otimes \mathbb{H}_q$$

$$\tilde{W} = V \oplus V \oplus W$$

$$\alpha = \begin{pmatrix} B_1 \otimes 1 - 1 \otimes x_{22} \\ B_2 \otimes 1 - 1 \otimes x_{22} \\ j \otimes 1 \end{pmatrix}$$

$$\beta = (-B_2 \otimes 1 + 1 \otimes x_{22},\ B_1 \otimes 1 - 1 \otimes x_{21},\ i \otimes 1)$$

$$\beta\alpha = 0 \iff [B_1, B_2] + ij = 0$$

$$(\zeta =) \beta\beta^* = \alpha^*\alpha \iff [B_1, B_1^*] + [B_2 B_2^*] + ii^* - j^*j = 0$$

α is injective, β is projective

$E = \ker\beta / \mathrm{Im}\,\alpha$ projective $\mathbb{H}_q$-module

$$\nabla : E \hookrightarrow \tilde{W} \otimes \mathbb{H}_q \xrightarrow{1 \otimes d} \tilde{W} \otimes \Omega^1_{\mathbb{H}_q} \xrightarrow{P \otimes 1} E \otimes \Omega^1_{\mathbb{H}_q}$$

$$P = 1 - \mathcal{D}^* \zeta^{-1} \mathcal{D}, \quad \mathcal{D} = \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix}$$

<u>Theorem</u>: (F-Jardim) The above construct.

$$(B_1, B_2\, i\, j) \to (E, \nabla)$$

maps the set of equivalence classes of ADHM data to gauge equivalence classes of quantum instantons

Note that our quantum instantons have "good" behavior at ∞, which is achieved by the similar const for $\mathbb{H}_q(\infty)$ and verification of the consistency under an identific

$$\mathbb{H}_q(0)[\delta^{-1}] \cong \mathbb{H}_q(\infty)[\delta]$$

We also expect (work in progress) that the above construction yields <u>all</u> gauge equivalence classes of quantum instantons.

To introduce Γ in the deformed case we need to find "finite subgroups" of $SU(2)_q$. Ostrik has classified them at the categorical level and obtained the classification by Dynkin diagrams A_n D_n $E_{6,7,8}$ and related it to classification of $\widehat{SU(2)}$ CFT by CIZ. It is an interesting problem (and is crucial for our construction) to find a direct link to $\Gamma \subset SU(2)$

3. Representation categories & Nakajima varieties

Nakajima geometrization of repres. theory of affine Lie algebras:

$\hat{\mathfrak{g}}$ - simply-laced affine Lie algebra

$\{V_{\hat{\lambda}}\}_{\hat{\lambda}\in\hat{P}_+}$ - integrable h.w. irred. repres.

$\hat{P}_+ = \bigoplus_{i=0}^{r} \mathbb{Z}_+\omega_i$; $\hat{Q}_+ = \bigoplus_{i=0}^{r} \mathbb{Z}_+\alpha_i$; $\langle\alpha_i, \omega_j\rangle = \delta_{ij}$

$V_{\hat{\lambda}} = \bigoplus_{\hat{\alpha}\in\hat{Q}_+} V_{\hat{\lambda}}(\hat{\lambda}-\hat{\alpha})$ - weight decompos

$\mathfrak{M}(\hat{\alpha},\hat{\lambda})$ - Nakajima varieties

$V_{\hat{\lambda}}(\hat{\lambda}-\hat{\alpha}) \cong H^{mid}(\mathfrak{M}(\hat{\alpha},\hat{\lambda}))$!!!

with geometric action of $\hat{\mathfrak{g}}$ on $V_{\hat{\lambda}}$

A version of Nakajima construction

is based on K-theory of $\mathcal{M}(\hat{\alpha}, \hat{\lambda})$, idea of Grojnowski

which yields "affinization" $\hat{\hat{\mathfrak{g}}}$ of $\hat{\mathfrak{g}}$.

It is natural to consider categories of vector bundles or coherent sheaves

$$\mathcal{C}(\mathcal{M}(\hat{\alpha}, \hat{\lambda}))$$

and consider "categorification" of representation theory of $\hat{\mathfrak{g}}$ & $\hat{\hat{\mathfrak{g}}}$.

<u>Conjecture:</u> The above categories can be identified with certain repres. categories for 3d current algebra on S^3/Γ, where $\Gamma \xleftrightarrow{\text{McKay}} \hat{\mathfrak{g}}$.

Nakajima varieties $\mathfrak{M}(\hat{\mathcal{L}},\hat{\lambda})$ can be viewed as resolutions of moduli spaces of instantons on ALE spaces which at ∞ look like S^3/Γ. By def

$$\mathfrak{M}'(\hat{\mathcal{L}},\hat{\lambda}) = \mathcal{A}(\hat{\mathcal{L}},\hat{\lambda})/\mathcal{G}(\mathcal{L},\hat{\lambda})$$

$\mathcal{A}(\hat{\mathcal{L}},\hat{\lambda})$ – ASDYM connections

$\mathcal{G}(\hat{\mathcal{L}},\hat{\lambda})$ – gauge group (4 dimensional)

Nekrasov – Schwarz ($\Gamma = e$) and Lazario (all Γ) found noncommut. deformation of ALE spaces that yields directly $\mathfrak{M}(\hat{\mathcal{L}},\hat{\lambda})$.

Heuristic construction of vector bundles
(inspired by G. Segal: Faddeev's anomaly preprint)

$H_A = \Lambda(V_- \oplus V_+^*)$, where polarization

is constructed using $D_A = D + A$, $A \in \mathcal{A}(\hat{\alpha}, \hat{\lambda})$

$\mathcal{H} \overset{def}{=} \bigcup_{A \in \mathcal{A}(\hat{\alpha}, \hat{\lambda})} H_A$ – bundle of Cliff mod

$\mathcal{P} \overset{def}{=} \bigcup_{A \in \mathcal{A}(\hat{\alpha}, \hat{\lambda})} P(H_A)$ – bundle of proj. sp.

$\mathcal{G} = \mathcal{G}(\hat{\alpha}, \hat{\lambda})$ acts on $\mathcal{P}$

$\hat{\mathcal{G}}$ an extension of $\mathcal{G}$ that acts on $\mathcal{H}$

$\mathcal{S}$ – space of sections of $\mathcal{H} \to \mathcal{A}(\hat{\alpha}, \hat{\lambda})$

$\mathcal{S} = \bigoplus_i S_i \otimes \mathcal{R}_i$, $\mathcal{R}_i$ irred rep. of $\hat{\mathcal{G}}$

S_i – vector bundles on $\mathcal{A}(\hat{\alpha}, \hat{\lambda})$

Recall the loop group extension

$\mathrm{Map}(S^1, G)^\wedge = \hat{\mathcal{G}}/\mathcal{G}_0$, where

$\mathcal{G} = \mathrm{Map}(\mathbb{R}^2, G)$ $\quad \lim_{r\to\infty} g(re^{i\theta})$ exist $\forall\theta$

$\mathcal{G}_0 \subset \mathcal{G}$ $\quad \lim_{r\to\infty} g(re^{i\theta}) = e \quad \forall\theta$

$1 \to S^1 \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$ given by 2-cocycle

$$\gamma(g_1, g_2) = \frac{1}{4\pi}\int_{\mathbb{R}^2} tr(g_1^{-1}dg_1 \wedge dg_2 g_2^{-1})$$

$\mathcal{G}_0 \hookrightarrow \hat{\mathcal{G}}$ $\quad g \to (g, e^{i\omega(g)})$, $\omega(g)$ WZ term

There is a similar construction

$\mathrm{Map}(S^3_\Gamma, G)^\wedge = \hat{\mathcal{G}}/\mathcal{G}_0$ (Faddeev/Mickelsson)

We expect that S_i corresponding to representations of $\mathrm{Map}(S^3/\Gamma, G)^\wedge$ yield a basis of $\mathcal{C}(\mathcal{M}(2,\hat{\lambda}))$

4. Categorification of representation theory of affine Lie algebras.

	Structure theory	Representation theory
1d	Extended Dynkin diagram = Groth. ring of Γ	Representation theory of $\hat{\mathfrak{g}}$
3d	Representation theory of Γ	Categorification of repres. theory of $\hat{\mathfrak{g}}$ = Representation theory of 3d current algebra

I Geometrization of rep. th. of $\hat{\mathfrak{g}}$ via Γ

= Groth. ring structure of rep. th. of 3d c.a.

II Categorification of rep. th. of $\hat{\mathfrak{g}}$ via Γ

= Representation category of 3d current alg.

Category of Γ-mod or $\mathbb{C}[\Gamma]$-mod is semisimple, not good for our goal

Natural pair of Koszul dual algebras

$$S_\Gamma = S(\rho) \rtimes \mathbb{C}[\Gamma], \quad \Lambda_\Gamma = \Lambda(\rho) \rtimes \mathbb{C}[\Gamma]$$

$$V \in S_\Gamma\text{-mod}, \quad V|_\Gamma = \bigoplus_{i=0}^{r} V_i \otimes \rho_i$$

$$\underline{\dim}\, V = \sum_{i=0}^{r} \dim V_i \, \alpha_i \in \hat{Q}_+$$

Homological presentation of the Bilinear form in $\hat{Q}$ (Crawley-Boevey)

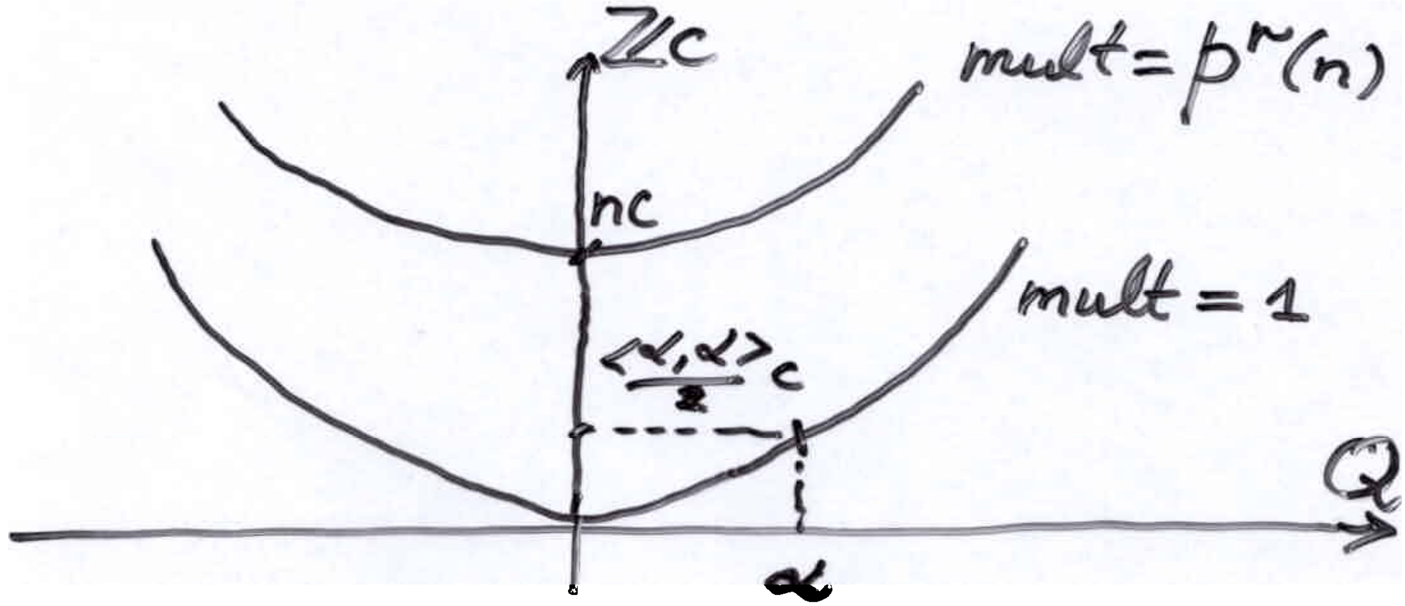
$$\langle \underline{\dim}\, V, \underline{\dim}\, W \rangle = \dim \mathrm{Hom}(V, W) -$$

$$- \dim \mathrm{Ext}^1(V, W) + \underbrace{\dim \mathrm{Ext}^2(V, W)}_{\parallel \atop \dim \mathrm{Hom}(W, V)}$$

$$\mathrm{Ext}^i(V, W) = 0 \qquad i > 2$$

Example: Basic representation $\hat{\lambda}=\omega_0$

$$V_{\omega_0} = \bigoplus_{\hat{\alpha}\in\hat{Q}_+} V_{\omega_0}(\omega_0-\hat{\alpha})$$



Affine Weyl group action

$s_i\bullet\hat{\alpha} \overset{\text{def}}{=} s_i(\hat{\alpha}-\omega_0)+\omega_0$, then

$s_i\bullet\hat{\alpha} = \hat{\alpha}+(\delta_{0,i}-\langle\hat{\alpha},\alpha_i\rangle)\alpha_i$, since

$\langle\omega_0,\alpha_i\rangle=\delta_{0,i}$

$$s_i : V_{\omega_0}(\omega_0-\hat{\alpha}) \xrightarrow{\sim} V_{\omega_0}(\omega_0-s_i\bullet\hat{\alpha})$$

What are the varieties $\mathfrak{M}(\hat{\alpha}, \omega_0)$?

$\mathfrak{M}(\hat{\alpha}, \omega_0) \cong$ moduli space of cyclic S_Γ modules of dimension $\hat{\alpha}$,

i.e. for $V \in \mathfrak{M}(\hat{\alpha}, \omega_0)$

$$\exists v_0 \in V : \quad \Gamma v_0 = v_0 , \quad S_\Gamma v_0 = V$$

Let $c = \sum_{i=0}^{r} \dim \rho_i \, \alpha_i$ isotropic vector in $\hat{Q}$

$\mathfrak{M}(c, \omega_0) \cong \widetilde{\mathbb{C}^2/\Gamma}$ Kronheimer

$\mathfrak{M}(nc, \omega_0) \cong \widetilde{\mathbb{C}^2/\Gamma}[n]$ Nakajima

Equivariant Hilbert schemes

diffeomorphic to actual Hilbert schemes

General varieties $\mathfrak{M}(\hat{\alpha}, \omega_0)$ are reduced to the equivariant Hilbert schemes $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$ by means of reflection functors

<u>Theorem:</u> There exist equivalences

$$\sigma_i : \mathfrak{M}(\hat{\alpha}, \omega_0) \xrightarrow{\sim} \mathfrak{M}(s_i \cdot \hat{\alpha}, \omega_0), \quad i = 0, \dots, r$$

and natural transformations

$$\sigma_i^2 \approx id$$

$$\sigma_i \sigma_j \approx \sigma_j \sigma_i \qquad \langle \alpha_i, \alpha_j \rangle = 0$$

$$\sigma_i \sigma_j \sigma_i \approx \sigma_j \sigma_i \sigma_j \qquad \langle \alpha_i, \alpha_j \rangle = -1$$

<u>Remark:</u> $\exists$ description of reflection functors in terms of Γ $\forall \hat{\lambda}$ (joint work with Khovanov and Schiffman)

Construction of reflection functors

(inspired by Crawley-Boevey AJM 122 (2000))

Homological formula implies

$$\delta_{0,i} - \langle \hat{\alpha}, \alpha_i \rangle = \underbrace{\dim \mathrm{Ext}^1(M, \rho_i)}_{\overset{\text{def}}{=}\, d_1} - \underbrace{\dim \mathrm{Hom}(\rho_i, M)}_{\overset{\text{def}}{=}\, d_2}$$

where $M \in \mathfrak{M}(\alpha, \omega_0)$, since

$$\dim \mathrm{Hom}(M, \rho_i) = \begin{cases} 0 & i = 1, \ldots, r \\ 1 & i = 0 \end{cases}$$

Then $\sigma_i(M) \in \mathfrak{M}(s_i \cdot \alpha, \omega_0)$ is determined

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \rho_i^{d_2} \arrow[d] & \rho_i^{d_2} \arrow[d] & \\
0 \arrow[r] & \rho_i^{d_1} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \rho_i^{d_1} \arrow[r] & \sigma_i(M) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Theorem implies

$$\mathfrak{M}(\alpha + mc, \omega_0) \cong \mathfrak{M}(nc, \omega_0), \quad n = m - \frac{\langle \alpha, \alpha \rangle}{2}$$

We want to describe $K(\mathfrak{M}(nc, \omega_0))$

$$K(\mathfrak{M}(c, \omega_0)) \cong K(Rep(\Gamma))$$

Gonzalez-Sprinberg - Verdier, Ito-Nakam
Ito-Nakajima, ura,

$$\widetilde{\mathbb{C}^2/\Gamma} \cong \left((\mathbb{C}^2)^{[N]}\right)^{\Gamma}_{main}, \quad N = |\Gamma|$$

$$S = \bigoplus_{i=0}^{\Gamma} S_i \otimes \rho_i \qquad \text{tautological vect. Bund}$$

Then $\{S_i\}_{i=0}^{\Gamma}$ form a basis of $K(\mathfrak{M}(c, \omega_0))$

$$K(\mathfrak{M}(nc, \omega_0)) \cong K(Rep(\Gamma_n)): \qquad \text{Wang's conjecture}$$

$$\Gamma_n \overset{def}{=} (\underbrace{\Gamma \times \dots \times \Gamma}_{n}) \rtimes S_n$$

$(\mathbb{C}^2)^{[n]} \cong S_n$-Hilbert scheme : Haiman

Thus we can identify (mod Wang's conjecture)

$$\bigoplus_{\hat{\alpha}\in\hat{Q}_+} K(\mathfrak{M}(\hat{\alpha},\omega_0)) \cong \Big(\bigoplus_{n\geq 0} K(\mathrm{Rep}(\Gamma_n))\Big)\otimes \mathbb{C}[\underbrace{K(\mathrm{Rep}(\Gamma))}_{=\hat{Q}}]$$

Construction of basic representation via Γ

<u>Theorem</u> (F-Jing-Wang):

There is an isomorphism

$$\Big(\bigoplus_{n\geq 0} K(\mathrm{Rep}(\Gamma_n))\Big)\otimes\mathbb{C}[K(\mathrm{Rep}(\Gamma))] \cong$$

$$S(\hat{\mathfrak{h}}(-1)\oplus\hat{\mathfrak{h}}(-2)\oplus\ldots)\otimes\mathbb{C}[\hat{Q}]$$

where $\hat{\mathfrak{h}}=\hat{Q}\otimes\mathbb{C}$, and a represent of double affine Lie algebra $\hat{\hat{\mathfrak{g}}}$ is constructed using induction and restriction $\Gamma_m\times\Gamma_n \longleftrightarrow \Gamma_{m+n}$

Remarks:

1) Note that F-Kac/Segal construct.

$$V_{\omega_0} \cong S(\mathfrak{h}(-1) \oplus \mathfrak{h}(-2) \oplus \dots) \otimes \mathbb{C}[Q]$$

admits natural "affinization" in geometric setting

2) The original construction of V_{ω_0} using Hilbert schemes $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$ has been obtained by Grojnowski/Nakajima

3) Categorification of the basic repres. can be achieved only by replacing $\mathbb{C}[\Gamma_n]$ by $S(\rho^{\otimes n}) \rtimes \mathbb{C}[\Gamma_n]$. Note that their K-theories are isomorphic. (F-Khovanov-Malkin, in progress)

Combining our conjectures in rank 1 we conclude that representation theory of $\mathrm{Map}(S^3/\Gamma, \mathbb{H}^{1\!1})^\wedge$ should have a representation category whose Grothendieck ring is given by affinization of vertex representations

$$S(\hat{\mathfrak{h}}(-1) \oplus \hat{\mathfrak{h}}(-2) \oplus \dots) \otimes \mathbb{C}[\hat{Q}]$$

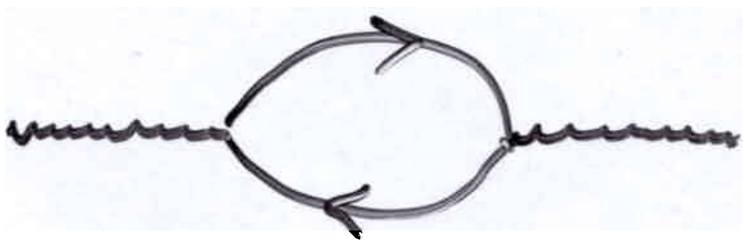
In particular, for $\Gamma = e$ we get the space of the boson-fermion corres.

$$S(h_{-1}, h_{-2}, \dots) \otimes \mathbb{C}[\mathbb{Z}]$$

Conclusion

Relation to physics : 4d QFT
cf. affine Lie algebras repres and 2d QFT

1) Divergence of cocycle for $\widehat{Map(S^3, \mathbb{H})}$ is related to polarization of vacuum



2) Quantum deformation with classical moduli corresponds to Penrose' idea of primary role of lightcone rays

3) ALE spaces $\widetilde{\mathbb{C}^2/\Gamma}$ are known as gravitational instantons. They are simplest manifistation of quantum gravity