A. Voronov.

Semisimplicity of orbifold vertex algebras

Conjecture : The orbifold V^G of a rational VOA V is itself rational (G - finite group)

— Joint work with Bakalov and Kac

Theorem (Dong, Li, Mason) : If a VOA is semisimple, then it is rational.

Theorem 1. If V is weakly semisimple, then V^G is weakly semi-simple.

Theorem 2. If V is a semisimple conformal assoc. algebra, then V^G is also semisimple.

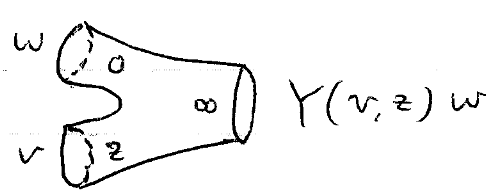
---

1. VOAs.

- V vector space
- $V \to \operatorname{End} V[[z, z^{-1}]]$

$v \mapsto Y(v, z)$

$$Y(v, z) w = \sum_{j \in \mathbb{Z}} v_j w \, z^{-j-1}$$



VOA - module M.

$$V \longrightarrow \mathrm{End}\, M[[z, z^{-1}]]$$
$$v \longmapsto Y_M(v, z).$$

2. Semisimplicity

Def 1. A VOA is $\underbrace{\text{semisimple}}^{\text{weakly}}$, if it is completely reducible as a module over itself, i.e. every ideal admits a direct complement as a module.

2. A VOA is semisimple, if its category of (weak) modules is semisimple.

Orbifold: G compact group acting on a VOA by automorphisms.

$$V^G = \text{fixed pts}, \text{ is a VOA} \subset V.$$

Question: If V is (weakly) semisimple, will V^G be (weakly) semi-simple? (for G finite).

Theorem 1 V weakly semi-simple
$\Longrightarrow V^G$ is weakly semi-simple.

(This follows from Theorem 1')

$\Leftarrow$ <u>Theorem 1'</u>. If V is weakly semi-simple,

and $V = U \oplus M$ as U-modules.

U: VOA ~~submodule~~ subalgebra; M: U-module.

then U is weakly semi-simple.

- <u>Thm 1' $\Rightarrow$ Thm 1</u>

$$V = V^G \oplus M$$

$$M = \bigoplus_{\substack{\lambda \in \widehat{G} \\ \lambda \neq 1}} \left(V^\lambda \otimes M_\lambda \right)$$

M_λ: corr. rep of G.

- Proof of <u>thm 1'</u>.

Suppose ideal $I \triangleleft U$.

Take $VI \triangleleft V$. Then by weak semisimplicity of V, we have splitting $V \to VI \to 0$ as V-modules.

$$VI = (U \oplus M) I = I \oplus MI$$

$$V \to VI \to I \to 0 \quad \text{a splitting projection for } I \subset U.$$

$$V \supset U$$

$$\Rightarrow U = I \oplus M' \quad \blacksquare$$

Ψ-theorem If V is semisimple and G compact, then V^G is semisimple

Ψ-proof:
$$\mathrm{Ext}^1_U(M,N) = H^1(U, \mathrm{Hom}_{\mathbb{C}}(M,N))$$
$$\overset{\text{Shapiro}}{=} H^1(V, \mathrm{Hom}_{U\otimes U^{op}}(V\otimes V^{op}, \mathrm{Hom}(M,N)))$$

Cohomology $\dfrac{\text{VOA} \quad \text{Kimura-AV}}{\text{Conf. alg.} \quad \text{Bakalov-Kac-AV}}$
$$= H^1(V, \mathrm{Hom}(V\otimes_U M, V^*\otimes_U N))$$
$$\cancel{= \mathrm{Ext}^1_V(V\otimes_U M, V^*\otimes_U N) = 0}$$

Remark: The Ψ-Theorem is known to be not true. **Exercise**: find errors in Ψ-Proof.

Conformal algebras

An *associative conformal algebra* is a $\mathbb{C}[\partial]$-module V with

data
$$V\otimes V \longrightarrow V[\lambda]$$
$$a\otimes b \longmapsto a_\lambda b = \sum_{j\geq 0} \lambda^{(j)} a_j b$$

axioms
$$(\partial a)_\lambda b = -\lambda a_\lambda b$$
$$a_\lambda(\partial b) = (\partial+\lambda) a_\lambda b$$
$$a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$$

M a conformal V-module
$$V\otimes M \longrightarrow M[\lambda]$$
$$a_\lambda m = \sum \lambda^{(j)} a_j m$$

G compact group of automorphisms of V.

$\rightsquigarrow V^G$ orbifold.

<u>Theorem 2</u> If V is semisimple, then V^G is so.

$\Leftarrow$ <u>Theorem 2'</u>: If $V = U \oplus M$ is semisimple then U is so.

<u>Proof</u>: Suppose $N_1 \xrightarrow{f} N_2 \to 0$ exact sequence of U-modules

Want to show it splits.

<u>Induced module</u>:

$$(V \otimes N)[\lambda] = \{ v \otimes_\lambda n := \sum_{\substack{j \geq 0 \\ \text{finite}}} v \otimes_j n \lambda^{(j)} \}$$

$$V \otimes_U N = V \otimes N [\lambda] \Big/ \quad v \otimes_\lambda (u \otimes_\mu n) \sim (v_\lambda u) \otimes_{\lambda+\mu} n.$$

$v \in V, u \in U, n \in N$

It is a V-module

Get $V \otimes_U N_1 \xrightarrow{id \otimes f} V \otimes_U N_2 \to 0$

$\to$ splits because V is semisimple.

Also $V = U \oplus M$, so

$$N_1 \oplus M \otimes_U N_1 \underset{s}{\overset{id \otimes f}{\rightleftarrows}} N_2 \oplus M \otimes_U N_2 \to 0$$

$$id_V \otimes f = f|_{N_1} + id_M \otimes f|_{N_1}.$$

$\boxed{f|_{N_1} = f}$

$$(f|_{N_1} \oplus id_M \otimes f|_{N_1})s = id$$

$$f|_{N_1} s + (id_M \otimes f|_{N_1}) s = id_{N_1} + id_{M \otimes_U N_1}$$

$f|_{N_1} s = id_{N_1}$ gives a splitting of f as U-modules. ∎