Duality $\hbar \to \frac{1}{\hbar}$ which has been already discussed in Joerg Teschner talk. He considered $SL(2,\mathbb{R})$. I want to consider some other algebraic varieties. My real goal is to show that $\hbar \leftrightarrow \frac{1}{\hbar}$ corresponds to forms $\leftrightarrow$ cycles

For reference: in my talk $\gamma = \pi\hbar$

Techer β : $\gamma = \beta^2$

<u>Nakayashiki</u>

Algebraic integrable models

Consider $\mathbb{C}^N$, $A_0 = \mathbb{C}[x_1 \dots x_N]$

$f_j(x_1 \dots x_N)$ polynomials

$$\mathcal{M}_{2g}^{\mathbb{C}} = \{ x \in \mathbb{C}^N \mid f_j(x_1 \dots x_N) = 0 \}$$

Complex dimension $2g$ (g - genus for future).

$$A = A_0 / \{ f_j = 0 \}$$

f_j are in the center

Poisson brackets on $\mathbb{C}^N$ (quadratic)

Integrability $\exists\, u_1 \dots u_g : \{u_i, u_j\} = 0$

$U = \mathbb{C}[u_1 \dots u_g]$ linear in our case

Consider the real form $\mathbb{C}^N \to \mathbb{R}^N$

Level of integrals

$$\mathcal{M}_{2g}^{\mathbb{R}}(u^{(0)}) = \{ x \in \mathcal{M}_{2g}^{\mathbb{R}} \mid u_j(x) = u_j^{(0)} \}$$

Real torus $T_g^{\mathbb{R}}$

Locally $\varphi_1 \dots \varphi_g \quad J_1 \dots J_g$

$\forall a \in A^{\mathbb{R}}$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T a(t)dt = \int_{\mathcal{M}_{2g}^{\mathbb{R}}(u^{(0)})} a(J,\varphi)\, d\varphi_1\wedge\dots\wedge d\varphi_g$$

if the original point of trajectory belongs to $\mathcal{M}_{2g}^{\mathbb{R}}(u^{(0)})$.

Top form on the torus. Nothing to say?

~~Glob~~ <u>Algebro geometric approach to cohomologi</u>

$$D_j a = \{u_j, a\}$$

$$[D_i, D_j] = 0$$

dt_j dual to D_j

Diff forms $\quad a_{i_1\dots i_k}\, dt_{i_1}\wedge\dots\wedge dt_{i_k} \qquad C^k$

$$d = \sum D_j\, dt_j = \sum \frac{\partial}{\partial\varphi_j} d\varphi_j \quad d^2 = 0$$

$$D_j u_k = 0, \qquad D_j = \sum c_{jk}\frac{\partial}{\partial\varphi_k}$$

We can put $\to dt_1\wedge\dots\wedge dt_j = C(J)\, d\varphi_1\dots\wedge d\varphi_j$

$$C^0 \xrightarrow{d} C^1 \rightarrow \ldots \xrightarrow{d} C^g$$

$H^k(C^*, F)$ finite dimensional

if H^g is represented as

$$v_\alpha \, dt_1 \wedge \ldots \wedge dt_g \qquad \alpha = 1, \ldots, b_g$$

$$a = \sum_{\alpha=1}^{b_g} P_\alpha(D_1 \ldots D_g, u_1 \ldots u_g) \, v_\alpha$$

A is finitely generated as D-module

Algebro-geometrical meaning of cohomologies

$\mathcal{M}_{2g}^{\mathbb{C}}(u^{(0)})$ is g-dimensional affine variety. Actually $T_g^{\mathbb{C}} - \mathcal{D}$, moreover

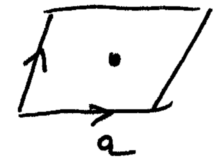
$D = \Theta$. $T_g^{\mathbb{C}} = J$

$H^0, \ldots H^g$

These cohomologies are the same as defined above.

On this complex variety we have dual homologies $H_0 \ldots H_g$

$\mathcal{M}^R_{2g}(u^{(0)})$ is one of g-cycles $\quad g = 1$



The mean value

$$\int\dots\int a(\varphi, J)\,d\varphi_1\dots d\varphi_g = \int_{\mathcal{M}^R_{2g}(u^{(0)})} \omega$$

~~where are other integrals~~

Cohomologies define different transcenden quantaties. But still where are

$$\int_c \mu \ ?$$

Quantization.

$q = e^{i\gamma}$ $\quad$ γ is the Plank constant

$A(q)$ $\quad$ $A(1) = A$, Poisson structure

commuting $u_1 \dots u_g$

$Q = e^{\frac{\pi^2 i}{\gamma}}$ $\quad$ $A(Q)$

Real form in classics $\leftrightarrow$ Hilbert space

"Example" $u = e^{2i\frac{d}{dx}}$, $v = e^{\gamma x}$

$$U = e^{\frac{2\pi}{\gamma} i \frac{d}{dx}}, \quad V = e^{\pi x}$$

$$uv = q^2 uv \qquad UV = Q^2 VU$$

One has to be careful with these very unbounded operators

$$[A(q), A(Q)] = 0$$

$u_1 \dots u_g \quad U_1 \dots U_g$

Spectrum $|\psi\rangle$, the spectrum is simple!

$\langle \psi | a A | \psi \rangle \qquad u_j |\psi\rangle = u_j^{(0)} |\psi\rangle \qquad U_j |\psi\rangle = U_j^{(0)} |\psi\rangle$

$a \in A(q)$, $A \in A(Q)$

~~Quasi classical limits $q \to 0$, $Q \to \infty$~~

Similarly to classical case

$$a = \sum \cancel{P_\alpha^L(u)\, v_\alpha\, P_\alpha^R(u)}\; S_\alpha(\overleftarrow{u}, \overrightarrow{u})\, v_\alpha$$

$$A = \sum \cancel{P_\alpha^L(U)\, V_\alpha\, P_\alpha^R(U)}\; S_\alpha(\overleftarrow{U}, \overrightarrow{U})\, V_\alpha$$

$\langle\psi| u_\alpha V_\beta |\psi\rangle$

Two classical limits $\gamma \to 0$, $\gamma \to \infty$

$\gamma \to 0 \qquad u_\alpha \to u_\alpha \, dt_1 \wedge \cdots \wedge dt_g = \mu_\alpha$

$\exists$ way $V_\beta \to C_\beta$

$$\langle\psi| u_\alpha V_\beta |\psi\rangle \xrightarrow[\gamma \to 0]{} \int_{C_\beta} \mu_\alpha$$

$$\langle\psi| u_\alpha V_\beta |\psi\rangle \xrightarrow[\gamma \to \infty]{} \int_{C_\alpha} \mu_\beta$$

In particular $V_0 = I$, $c_0 = \mathcal{M}^{\mathbb{R}}_{2g}(u^{(0)}$

The example that I have in mind

Classics

$$\ell(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

$a(z) = z^{g+1} + \cdots$

$b(z) = z^{g} + \cdots$

$d(z) = z^{g-1}\delta + \cdots$

$c(z) = z^{g}\gamma + \cdots$

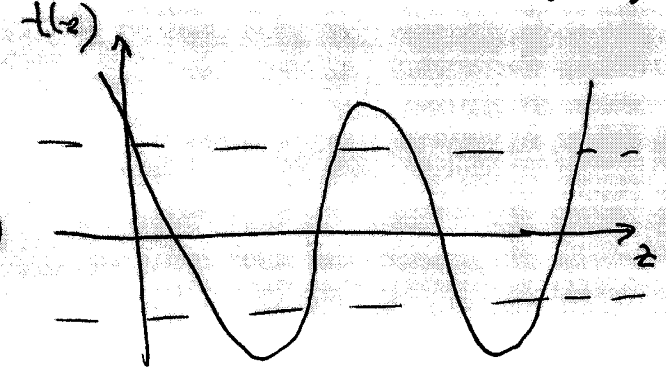
$\mathbb{C}^{4g+2}$, $\{,\}$

$\det \ell(z)$ is in center, let $\det \ell(z) = 1$

$t(z) \equiv \operatorname{tr} \ell(z) = z^{g+1} + \underbrace{u_1 z^{g} + \cdots + u_g z}_{} + c$

Remaining $2g$-dim

Real form a_j, b_j, c_j, d_j real

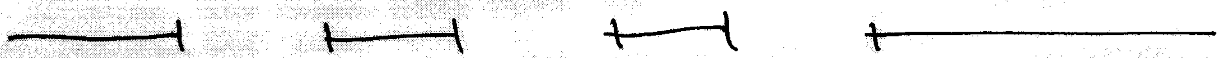


then $\dim(\text{level}) = g$

Spectral curve

$$\det |w - \ell(z)| = w^2 - t(z)w + 1 = 0$$

Hyper-elliptic curve



Holomorphic differentials

$$\sigma_k = \frac{z^{k-1}}{2w - t(z)}\, dz \qquad k = 1, \ldots, g$$

$$b(z) = \prod (z - z_j) \qquad w_j = d(z_j)$$

Canonical: $\{z_j, w_k\} = \delta_{jk}\, z_j w_j$

$$\ell(z) \to (p_1 \cdots p_g)$$

$$t_j = \sum \int_{p}^{p_k} \sigma_j + \Delta_j$$

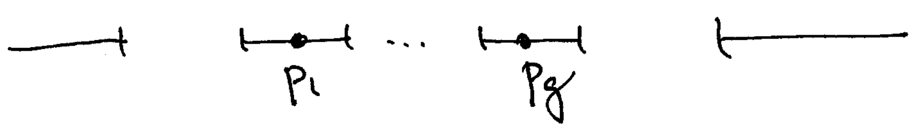
$j = 1, \ldots g$

$$\{u_j, t_k\} = \delta_{jk}$$

Level of integrals is real part of

$J - \Theta$

Dynamics



$$\int a\, d\varphi_1 \dots d\varphi_g = \int_{a_1} \cdots \int_{a_g} a(z,w) \prod_{i<j} (z_i - z_j) \prod \frac{dz_i}{2w_i - t_i}$$

Cohomologies are such that everything is expressable ~~in~~ in terms of

$$\int_{a_j} \frac{z^k}{2w - t(z)}\, dz \qquad b_k = \binom{2g}{k} - \binom{2g}{k-2}$$

~~Ithe~~ Oppositely: important property of h.e. Θ-divisor: ~~Ecomap~~

$H_k(J - \Theta) \subset H_k(J)$, for h.e. it is injective

Every cycle is a half-basis. Everything is expressable in terms of

$$\int_{c_j} \frac{z^k}{2w - t(z)}\, dz$$

Quantization $\qquad l(z)$

$$d(zq^2)a(z) - b(zq^2)c(z) = 1$$

$$[b(z), b(z')] = 0 \qquad b(z) = \prod (z - z_j)$$

$$[z_i, z_j] = 0.$$

$$d(z_j) = w_j \qquad w_j z_j = q^2 z_j w_j$$

$\mathcal{H} \simeq$ Space of functions of $\zeta_1 \dots \zeta_g$

$$\zeta_j = \frac{1}{2} \log z_j$$

Sklyanin:

$$|\psi\rangle = \prod Q(\zeta_j) \qquad w + w^{-1} = t(z)$$

$$Q(\zeta - i\gamma) + Q(\zeta + i\gamma) = t(z) Q(\zeta) \leftarrow$$ Like quantum dilogarithm

$Q(\zeta)$ - entire $\qquad t(z) = z^{g+1} + \dots$

$$Q(\zeta) \equiv Q_+(\zeta) + Q_-(\zeta)$$

$$Q_\pm(\zeta) \simeq e^{\left(-(1+\frac{\pi}{\gamma})\zeta \pm \frac{\zeta^2}{i\gamma}\right)(g+1)} (1+o())$$

Duality

$$W = Q\left(z + \tfrac{\eta+\gamma}{2}i\right) Q\left(z - \tfrac{\eta+\gamma}{2}i\right) - Q\left(z + \tfrac{\eta-\gamma}{2}i\right) Q\left(z - \tfrac{\eta-\gamma}{2}i\right)$$
$$= 1$$
$$\Downarrow$$

$$Q(z+\eta i) + Q(z-\eta i) = T(z)\, Q(z)$$

Mean values

$\langle \psi | a A | \psi \rangle$ are expressed in terms of

$$\int Q^2(z)\, z^k \otimes z^l \, dz$$

theire properties are magnificent!