Conformal Field Theory as a completely integrable quantum system.

V. V. Bazhanov (ANU, Canberra).

Review of joint work with

* S. Lukyanov & A.B. Zamolodchikov on "Integrable structure of CFT" (1994-2000)
* A. Hibberd & S. Khoroshkin (2002) "W_3 CFT".

IQFT in 2d

* Diagonalization of infinite set of local integrals of motion (LIM).

Infinite volume: spectrum of particles & S-matrix.

* CFT — simplest example of finite volume IQFT.

Continuous version of QIPM or Baxter's commuting transfer matrix method.

Plan

1. Integrale structure of CFT.

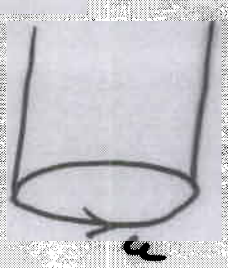
Applications:

2. Spectral theory of 1-d Schrödinger equation

3 Quantum Brownian motion in a periodic potential.

Local integrals of motion in CFT

$$T(u) = -\frac{c}{24} + \sum_{-\infty}^{\infty} e^{iun} L_{-n}$$



$-\infty < c < 1$

central charge

$$T(u) = T(u+2\pi)$$

Vir: $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m+n,0}$

Abelian subalgebra of $\mho$Vir

'88 Sasaki-Yamanaka
'89 Eguchi-Yang SK
'93 Feigin - Frenkel E

$[I_{2k-1}, I_{2l-1}] = 0$

$$I_{2k-1} = \int_0^{2\pi} T_{2k}(u)\frac{du}{2\pi}, \quad k = 1, 2, \ldots \infty$$

Local IM

$T_2(u) = T(u)$, $T_4(u) = :T^2(u):$, $T_6(u) = :T^3: + \frac{c+2}{12} :T'^2(u):$

($\deg T = 2$, $\deg \partial_u = 1 \Rightarrow \deg T_{2k} = 2k$.

$T_{2k}(u)$ is uniquely defined (up to total derivative)

$I_{2k-1} : \mathcal{V}_\Delta \to \mathcal{V}_\Delta$

$\mathcal{V}_\Delta : \quad L_{-n_1} L_{-n_2} \cdots |\Delta\rangle, \quad n_i > 0$

$L_n|\Delta\rangle = 0, \quad L_0|\Delta\rangle = \Delta|\Delta\rangle \quad n > 0.$

$I_1 = L_0 - \frac{c}{24}, \quad \Rightarrow \quad I_{2k-1} : \mathcal{V}_\Delta^{(l)} \to \mathcal{V}_\Delta^{(l)}$

$$L_0 \mathcal{V}_\Delta^{(l)} = (\Delta + l)\mathcal{V}_\Delta^{(l)}, \quad l = 0, 1, \ldots$$

Problem: diagonalize $\{I_{2k-1}\}$.

Appendix B.

Here we present eigenvalues of the local IM $\mathbf{I}_{2n-1}$ $(n = 1, 2, ..., 8)$ with $R = 2\pi$ on the highest Virasoro vector with the central charge c and conformal dimension Δ

$$I_1^{vac}(\Delta) = \Delta - \frac{c}{24}\,, \tag{B.1}$$

$$I_3^{vac}(\Delta) = \Delta^2 - \frac{(c+2)}{12}\,\Delta + \frac{c\,(5c+22)}{2880}\,, \tag{B.2}$$

$$I_5^{vac}(\Delta) = \Delta^3 - \frac{(c+4)}{8}\,\Delta^2 + \frac{(c+2)\,(3c+20)}{576}\,\Delta - \frac{c\,(3c+14)\,(7c+68)}{290304}\,, \tag{B.3}$$

$$\begin{aligned} I_7^{vac}(\Delta) = {}& \Delta^4 - \frac{(c+6)}{6}\,\Delta^3 + \frac{(15c^2+194c+568)}{1440}\,\Delta^2 - \frac{(c+2)\,(c+10)\,(3c+28)}{10368}\,\Delta \\ &+ \frac{c\,(3c+46)\,(25c^2+426c+1400)}{24883200}\,, \end{aligned} \tag{B.4}$$

$$\begin{aligned} I_9^{vac}(\Delta) = {}& \Delta^5 - \frac{5\,(c+8)}{24}\,\Delta^4 + \frac{(c+8)\,(5c+46)}{288}\,\Delta^3 \\ &- \frac{(35c^3+990c^2+9048c+23488)}{48384}\,\Delta^2 \\ &+ \frac{(c+2)\,(175c^3+7134c^2+96168c+392000)}{11612160}\,\Delta \\ &- \frac{c\,(5c+22)\,(11c+232)\,(7c^2+274c+1960)}{3065610240}\,, \end{aligned} \tag{B.5}$$

$$\begin{aligned} I_{11}^{vac}(\Delta) = {}& \Delta^6 - \frac{c+10}{4}\,\Delta^5 + \frac{15c^2+322c+1808}{576}\,\Delta^4 \\ &- \frac{105c^3+3700c^2+44612c+165984}{72576}\,\Delta^3 \\ &+ \frac{525c^4+27908c^3+548508c^2+4248784c+10147200}{11612160}\,\Delta^2 \\ &- \frac{(c+2)\,(315c^4+24604c^3+676548c^2+7298480c+25872000)}{418037760}\,\Delta \\ &+ \frac{c\,(13c+350)\,(11025c^4+1160780c^3+25741404c^2+198779728c+470870400)}{27389834035200}\,, \end{aligned} \tag{B.6}$$

Classical limit $c \to -\infty$

$$T(u) = -\frac{c}{6} U(u), \quad [\,,\,] \to \frac{6\pi}{ic}\{\,,\,\}$$

$$\{U(u), U(v)\} = 2\big(U(u) + U(v)\big)\delta'(u-v) + \delta'''(u-v).$$

Second Hamiltonian structure of KdV.

$$I_{2k-1} \longrightarrow I_{2k-1}^{(cl)}$$

$$I_1^{(cl)} = \int_0^{2\pi} U(u)\,du, \quad I_3^{(cl)} = \int_0^{2\pi} U^2(u)\,du, \dots$$

$$\partial_{t_{2k-1}} U(t_1, t_3, \dots) = \{I_{2k-1}^{(cl)}, U\}, \quad t_1 = u.$$

Isospectral deformation of the Schrödinger eq.

$$(\partial_u^2 + U(u))\Psi(u) = \lambda^2 \Psi(u).$$

$$T^{(cl)}(\lambda) = \mathrm{Tr}\, M(\lambda) \quad \text{monodromy matrix}$$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}(u+2\pi) = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}(u) M$$

$$U(u) = \phi'(u)^2 + \phi''(u), \quad \phi(u+2\pi) = 2\pi i p + \phi(u)$$

$$T_j^{(cl)}(\lambda) = \mathrm{Tr}_{\pi_j}\left(e^{2\pi i p H}\, \mathcal{P}\exp\left[\lambda \int_0^{2\pi} \left(e^{-2\phi(u)} E + e^{2\phi(u)} F\right) du\right]\right).$$

$\pi_j = (2j+1)$-dim rep. of sl_2 $\quad [E,F] = H, \; [H,E] = 2E, \; [H,F] = -2F.$

$$\{T_j^{(cl)}(\lambda), T_{j'}^{(cl)}(\lambda')\} = 0, \quad \{I_{2k-1}^{(cl)}, T_j^{(cl)}(\lambda)\} = 0$$

$$T^{(cl)}(\lambda) \equiv T_{1/2}^{(cl)}(\lambda).$$

$$\log T^{(cl)}(\lambda) \simeq 2\pi\lambda + \sum_{k=1}^{\infty} c_k\, I^{(cl)}_{2k-1}\, \lambda^{1-2k}, \quad \lambda \to \infty.$$

$T^{(cl)}(\lambda)$ – generating function for $I^{(cl)}_{2k-1}$

"Quantim KdV Theory".

We need to construct an appropriate CFT version of Baxter's commuting transfer matrix.

$$-gT(u) = :\varphi'(u)^2: + (1-g)\varphi''(u) + \frac{g}{24}$$ Feigin-Fuchs.

g: $\quad c = 13 - 6(g+g^{-1}),\quad \boxed{\begin{array}{c} -\infty < c < 1 \\ 0 < g < 1 \end{array}}$

$$\varphi(u) = iQ + iPu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu},$$

$$[P,Q] = \frac{ig}{2}, \quad [a_n, a_m] = \frac{n}{2} g\, \delta_{m+n,0}.$$

$\mathcal{V}_\Delta \sim \mathcal{F}_p$ (Fock module).

$P|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0, \quad n>0.$

$\Delta = \frac{p^2}{g} + \frac{c-1}{24}.$

Quantum Affine algebra $U_q(\widehat{sl}_2)$.

$$q = e^{i\pi g}, \quad \tilde{q} = e^{i\pi/g}$$

h, y_0, y_1 – generators of $\mathcal{B}_+ \subset U_q(\widehat{sl}_2)$

$[h, y_0] = 2y_0, \quad [h, y_1] = -2y_1$, (+ Serre relations)

$$T(\lambda) = \mathrm{Tr}\left[e^{2\pi i P h} \mathcal{P}\exp\left\{ \lambda \int_0^{2\pi} \left(e^{-2\varphi(u)} x_0 + e^{2\varphi(u)} x_1 \right) du \right\} \right]$$

1) "Evaluation" representations $\mathcal{B}_+ \to U_q(sl_2)$

$$h \to H^{(j)}, \quad x_0 \to q^{\frac{H^{(j)}}{2}} E^{(j)}, \quad x_1 \to q^{-\frac{H^{(j)}}{2}} F^{(j)}$$

$H^{(j)}, E^{(j)}, F^{(j)}$ — $(2j+1)$ dim rep. of $U_q(sl_2)$.

2) q-oscillator representations. $\mathcal{B}_+ \to H_q$

$$h \to \pm \mathcal{H}, \quad x_0 \to q^{\pm\frac{\mathcal{H}}{2}} \mathcal{E}_\pm, \quad x_1 \to q^{\mp\frac{\mathcal{H}}{2}} \mathcal{E}_\mp$$

$$H_q: \quad q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = 1, \quad [\mathcal{H}, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm .$$

① $T_j(\lambda)$ — quantum analog of $T_j^{(cl)}(\lambda)$.

② $$Q_\pm(\lambda) = \lambda^{\pm\frac{2P}{g}} \frac{\mathrm{Tr}[\quad]}{\mathrm{Tr}\left[e^{\pm 2\pi i P \mathcal{H}}\right]}.$$

CFT analogs of Baxter's Q-matrix.

$$[T_j(\lambda), T_{j'}(\lambda')] = [T_j(\lambda), Q_\pm(\lambda')] = 0 .$$

$$[T_j(\lambda), I_{2k-1}] = [Q_\pm(\lambda), I_{2k-1}] = 0 .$$

Properties

① $T_j(\lambda)$, $Q_\pm(\lambda)\cdot\lambda^{\mp\frac{2P}{g}}$ — entire functions of λ^2

② $\log T(\lambda) = c_0(g)\lambda^{\frac{1}{1-g}} + \sum_{n=1}^{\infty} C_n(g)\, I_{2n-1}\, \lambda^{\frac{1-2n}{1-g}}$

conjecture obtained from $\Delta\to\infty$. BLZ '96

$T(\lambda)\equiv T_{1/2}(\lambda)$

$$Q_\pm(\lambda,g,P) = \mathbb{C}(g,P)\, e^{\sum_{n=0}^{\infty} B_n(g)\, I_{2n-1}\, \lambda^{\frac{1-2n}{1-g}}} \quad *$$

$$\times\, Q_\pm\left(\lambda^{\frac{1}{g}}, g^{-1}, g^{-1}P\right).$$

$$Q_+(\lambda,g,P) = \lambda^{\frac{2P}{g}}\left(1+\sum_{n=1}^{\infty} a_n(g,P)\,\lambda^{2n}\right).$$

$$\boxed{\frac{Q_+(\lambda,g,P)}{Q_-(\lambda,g,P)} = \frac{Q_+(\lambda^{\frac{1}{g}}, g^{-1}, g^{-1}P)}{Q_-(\lambda^{\frac{1}{g}}, g^{-1}, g^{-1}P)}}$$

$Q_\pm(\lambda^2 e^{2\pi i}) = e^{\pm 2\pi i P} Q_\pm(\ldots$

③ $T(\lambda)\,Q_\pm(\lambda) = Q_\pm(\lambda q) + Q_\pm(\lambda q^{-1})$ Baxter's eqn.

④ $\underbrace{T_j(\lambda)}_{2i\sin(2\pi P)} = Q_+(\lambda q^{j+1/2})\,Q_-(\lambda q^{-j-1/2}) - Q_+(\lambda q^{-j-1/2})\,Q_-(\lambda q^{j+1/2})$

Quantum Wronskian rel.

⑤ $2i\sin(2\pi P) = Q_+(\lambda q^{1/2})\,Q_-(\lambda q^{-1/2}) - Q_+(\lambda q^{-1/2})\,Q_-(\lambda q^{1/2})$

$T_0(\lambda)\equiv 1.$

For any eigenvalue, the zeroes satisfy the Bethe-Ansatz eqn.

⑥ $\dfrac{Q_+(\lambda_k q)}{Q_+(\lambda_k q^{-1})} = -1$, $\quad Q_+(\lambda_k) = 0.$

$\beta^2 = g$

2

$$\begin{cases} \dfrac{Q_+(\lambda_k q)}{Q_+(\lambda_k q^{-1})} = -1 . \\ \lambda_k \sim k^{1-\beta^2} \qquad k \to \infty . \end{cases}$$

The Bethe Ansatz equations. Many solutions for different states in Virasoro module.

$E_n^{\pm}$, the spectrum of SE

$$\left(-\partial_x^2 + x^{2\alpha} + \frac{\ell(\ell+1)}{x^2}\right)\psi(x) = E\,\psi(x).$$

$$E_n^{+} = \rho\,\lambda_n^2 , \qquad \rho = \left(\frac{2}{\beta^2}\right)^{2-2\beta^2} \Gamma^2(1-\beta^2).$$

for Virasoro vacuum state.

$$\alpha = \frac{1}{\beta^2} - 1 ; \qquad \ell = \frac{2p}{\beta^2} - \frac{1}{2} \qquad \boxed{\beta^2 = g}$$

1) WHY? (Answer is not known).

2) How to prove this? (Answer is known).

Dorey & Tateo (Dec 1998.)

A. Voros 1990 ÷ 1998.

($\ell = 0$.) The similarity of the functional rels has been pointed out to me by Prof J-B Zuber. 1995

The harmonic oscillator

$$(-\partial_x^2 + x^2)\Psi(x) = E\,\Psi(x)$$

The discrete spectrum:

$$E_n = 2n+1, \qquad n = 0, 1, \ldots, \infty$$

The radial SE for the "anharmonic oscillator"

$$(-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha})\Psi(x) = E\,\Psi(x),$$

where $0 < x < \infty$. We consider $1 < \alpha < \infty$, but the results apply to $-1 < \alpha < \infty$. Solutions at small x. For $\mathrm{Re}\,\ell > -3/2$,

$$\psi(x, E, \ell) \simeq \sqrt{\frac{2\pi}{1+\alpha}}(2+2\alpha)^{-\frac{2l+1}{2+2\alpha}} \frac{x^{l+1}}{\Gamma(1+\frac{2l+1}{2+2\alpha})} + O(x^{l+3})$$

Basis of solutions (analytic continuation in ℓ)

$$\psi_+ = \psi(x, E, \ell), \qquad \psi_- = \psi(x, E, -\ell-1)$$

$$W[\psi_+, \psi_-] = 2i(q^{\ell+1/2} - q^{-\ell-1/2}),$$

$W[f,g] = fg' - f'g$ is the usua' Wronskian.

$$q = e^{i\pi g}, \qquad g = 1/(\alpha+1)$$

Spectral problem: for certain discrete values of E one these solutions will decay at $x \to \infty$,

$$E_n^+ : \quad \psi_+(x, E_n^+) \to 0 \;, \quad x \to \infty$$
$$E_n^- : \quad \psi_-(x, E_n^-) \to 0 \;, \quad x \to \infty$$

where $n = 1, \ldots, \infty$. Spectral determinants

$$D^{\pm}(E, l) = \prod_{n=1}^{\infty} (1 - \frac{E}{E_n^{\pm}}) \; .$$

Simple WKB analysis

$$E_n^{\pm} \sim n^{\frac{2\alpha}{1+\alpha}} \qquad \text{as } n \to \infty$$

For all E the SE has a unique solution which decays when $x \to \infty$

$$\chi(x, E, l) \to x^{-\frac{\alpha}{2}} \; \exp\{ - \frac{x^{1+\alpha}}{1+\alpha} + O(x^{1-\alpha})\}$$

This solution can be expanded in the basis $\{\psi_{\pm}\}$

$$\chi(x, E, l) = \frac{2\,[J_-(E)\psi_+(x, E) - J_+(E)\psi_-(x, E)]}{W[\psi^+, \psi^-]}$$

Jost functions

$$J_{\pm}(E,\ell) = D_{\pm}(E,\ell)$$

- entire functions of E with the leading asymptotics

$$\log D_{\pm}(E,\ell) \sim E^{(\alpha+1)/2\alpha}, \quad E \to \infty$$

- $D_{+}(E,\ell)$ is analytic function of ℓ for $\mathrm{Re}\,\ell > -3/2$

$$D_{-}(E,\ell) = D_{+}(E,-\ell-1)$$

Important symmetry of SE

$$\Omega: \qquad x \to qx, \qquad E \to q^{-2}E, \qquad \ell \to \ell$$

Define

$$\begin{aligned} \chi^{+}(x,E) &= \chi(x,E,l), \\ \chi^{-}(x,E) &= i\,q^{-\frac{1}{2}}\chi(qx,q^{-2}E,l) \end{aligned}$$

$$W[\chi^{+},\chi^{-}] = 2$$

$$\chi^- = \frac{2[q^{l+1/2}D_-(q^{-2}E)\psi_+ - q^{-l-1/2}D_+(q^{-2}E)\psi_-]}{W[\psi^+,\psi^-]}$$

"Quantum Wronskian" relation

$$q^{l+\frac{1}{2}}\,D_+(E)\,D_-(q^{-2}E) - q^{-l-\frac{1}{2}}\,D_+(q^{-2}E)\,D_-(E)$$

$$= q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}}\,,$$

This functional relation together with the analytic properties in E and ℓ uniquely define $D_\pm(E)$ and thus the spectrum of the SE.

$$D_\pm(E) = \lambda^{\mp\frac{2p}{q}}\,Q_\pm^{(vac)}(\lambda)$$

$$E \sim \lambda^2$$

Equations for E_n^+

$$\frac{D_+(q^2E^+n)}{D_+(q^{-2}E^+n)} = -q^{2\ell+1}, \quad \forall n$$

or

$$\prod_{k=1}^{\infty}\left(\frac{E_k^+ - q^2\,E_n^+}{E_k^+ - q^{-2}\,E_n^+}\right) = -q^{2\ell+1}$$

Quantum Brownian motion

The Brownian motion of a classical particle is described by the Langevin eq.

$$M\ddot{X} + \eta\dot{X} + \frac{\partial U(X)}{\partial X} = \xi(t)$$

where $\xi(t)$ is the random force

$$< \xi(t)\xi(t') >= 2\eta T\delta(t-t')$$

A simplest quantum model which reduces to the Langevin eq. at high temperatures $T \to \infty$ is the Caldeira-Leggett model. In this model the particle co-ordinate X is coupled to a "bath" of an infinite set of oscillators which provide both the friction and the fluctuating force.

$$\mathcal{H} = \frac{P^2}{2M} + U(X) + \sum_a \frac{p_a^2}{2m_a} + \frac{1}{2}\sum m_a\omega_a^2(x_a + \frac{\lambda_a X}{m_a\omega_a^2})^2$$

"Weighted density of states"

$$J(\omega) = \frac{\pi}{2}\sum_a \frac{\lambda_a^2}{m_a\omega_a}\delta(\omega - \omega_a)$$

Ohmic dissipation (infinite friction limit)

$$J(\omega) = \eta\omega$$

$$U(X) = \kappa \cos(X) - VX$$

V — driving force. The problem is to calculate the mobility

$$J = < \dot{X} >$$

(average drift velocity).

Applications:

Small size Josephson junctions DC current

Point contact current in fractional Quantum Hall Effect.

$X \sim$ angle variable $0 < X \leq 2\pi$

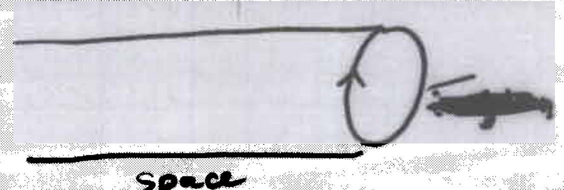
Field Theory model:

$$\mathbf{H} = \frac{1}{4\pi g}\int_{-\infty}^{0} dx(\Pi(x)^2 + \Phi_x^2) - \frac{\kappa}{g}\cos(\Phi_B + Vt)$$

canonical commutation relations

$$[\Pi(x), \Phi(x)] = -2i\pi g\delta(x - x')$$

and $\Phi_B = \Phi(0)$.



$$J = \langle \partial_x \phi \rangle$$

The answer for J is

$$J(V) = V - 2\pi i T E \partial_E \log S(E, \ell)$$

where

$$\ell = -1/2 + iV/2\pi T, \quad E = -\frac{\kappa^2 \pi}{\Gamma(1+g)^2} (\pi T)^{2g-2}$$

$$S(E, l) = -\frac{D_+(E, \ell)}{D_-(E, \ell)}$$

Duality $g \to 1/g$

$$\left(-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha}\right)\Psi(x) = E\,\Psi(x),$$

Change of variables

$$x = e^y, \quad \Psi(x) = e^{\frac{y}{2}}\tilde{\Psi}, \quad \nu = i(\ell + 1/2)$$

$$-\partial_y^2\,\tilde{\Psi} + \{e^{\frac{2y}{g}} - E\,e^{2y}\}\,\tilde{\Psi} = \nu^2\,\tilde{\Psi},$$

$$\tilde{\Psi}(y) \to e^{i\nu y} + S(\nu, E)\,e^{-i\nu y} \quad y \to -\infty,$$

$$J(V, T, g) = -V - g^{-1} J(gV, T, g^{-1})$$

Schmid '83, Fendley Ludwig Saleur '95 - duality at $T=0$.
Fisher (MPA), Zweger '85 conjectured duality at $T \neq 0$.
Fendley, Ludwig, Saleur '95 calculated current J
for $g = \frac{1}{N}$ from TBA approach.