Ideals in the Weyl algebra

George Wilson

Imperial College

The Weyl algebra A

$$A = \mathbb{C}\langle x, y\rangle / (xy - yx - 1)$$

$$= \mathbb{C}[x, \partial/\partial x] = \mathcal{D}(\mathbb{A}^1).$$

Problem : describe the space (?)

$$\mathcal{R} = \{\text{right ideals of } A\} / pI \sim qJ$$

$$= \{\text{rank 1 t.f. f.g } A\text{-modules}\} / \sim$$

$$= \{\text{line bundles over } \mathbb{A}^2_q\}.$$

Plan

1. Solution of LeBruyn$^{\sim}$ (1994)
2. Solution of Cannings – Holland (").
3. Connection between 1 and 2 ($\int$ble systems 1978, G.W. 1998)
4. The group of automorphisms of A (Yu. Berest – W., recent).

Commutative analogue

Weyl $A \rightsquigarrow A_0 = \mathbb{C}[x, y]$;

$$R_0 = \{\text{ideals in } A_0\} / pI \sim qJ$$

$$= \coprod_n \{\text{ideals of codimension } n\}$$

$$= \coprod_n \mathrm{Hilb}_n(\mathbb{A}^2)$$

$$= \{(X, Y, v)\} / \sim$$

where X, Y are commuting $n \times n$ matrices, v is a cyclic vector.

[Exercise: find a similar treatment in the NC case]

<u>Le Bruyn idea</u> : think of an ideal $M \subseteq A$ as a sheaf over $\mathbb{A}^2_q$ and extend it to a sheaf $\mathcal{M}$ over $\mathbb{P}^2_q$. Then imitate the usual classification of bundles over $\mathbb{P}^2$ (Barth, Beilinson, ...).

<u>Main difficulty</u> is to understand what is $\mathbb{P}^2_q$. More generally, what is NC algebraic geometry? (M. Artin, J. Zhang, M. van den Bergh ...).

Crash course in NC projective geometry

Commutative case.

$$X \subseteq \mathbb{P}^N \longleftrightarrow \underline{A}_X = \mathbb{C}[x_0, \dots, x_n]/I$$

(graded ring, Noetherian, connected ...).

$$\text{Sheaf } \mathcal{M}/X \longrightarrow \bigoplus_{k \geq 0} H^0(\mathcal{M}(k)),$$

a graded $\underline{A}_X$-module.

Serre theorem (FAC)

$$\mathrm{coh}(X) \xrightarrow{\cong} \mathrm{gr\,mod}\,\underline{A}_X / \sim,$$

where $\sim$ kills f.d. modules.

<u>NC case</u> <u>Start</u> with $\underline{A}$ (graded Noetherian, ...) and define:

$$X = \mathrm{proj}(\underline{A})$$

$$\mathrm{coh}(X) = \mathrm{gr\,mod}\,\underline{A}/\sim$$

$$\mathcal{O}_X = \pi(\underline{A})$$

$[\pi : \mathrm{gr\,mod}\,\underline{A} \longrightarrow \mathrm{coh}(X)]$

$$H^i(X, \mathcal{M}) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{M})$$

etc. etc. Under mild assumptions on $\underline{A}$, can prove Serre theorems:

- $\dim H^i(X, \mathcal{M}) < \infty$
- $H^i(X, \mathcal{M}(k)) = 0$ for $i > 0$, $k \gg 0$
- duality.

Our case: $\mathbb{P}^2_q \stackrel{\text{df}}{=} \text{Proj}\, \underline{A}$, where

$$\underline{A} = \mathbb{C}[X, Y, Z], \quad [X, Y] = Z^2.$$

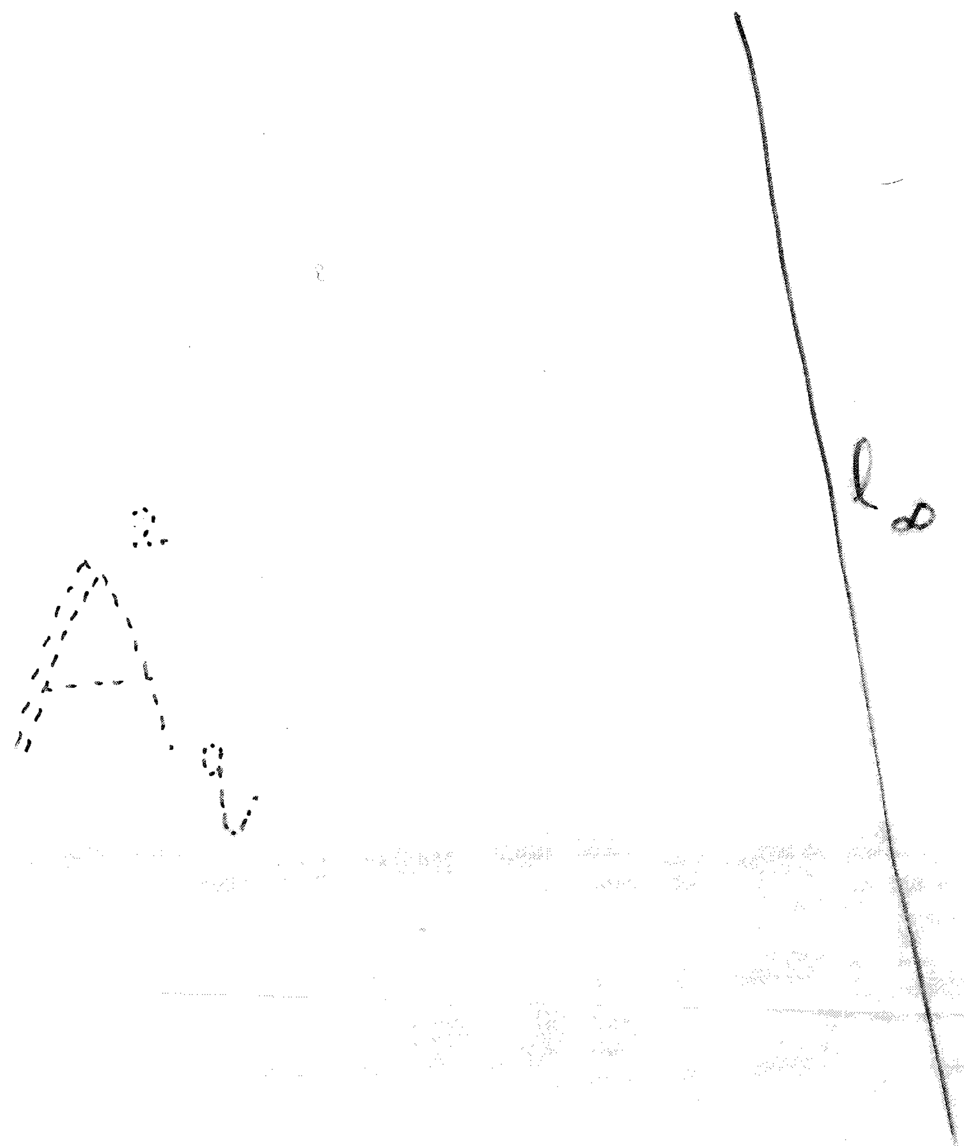
Note $Z = 1 \implies A$ (Weyl)

$Z = 0 \implies \mathbb{C}[X, Y] \leftrightarrow \mathbb{P}^1$

(affine plane + line at ∞).

[Crash course (ctd): there are lots of "NC $\mathbb{P}^2$"s (classified by Artin, Schelter, v.d. Bergh ...) They are classified by (roughly) their set of points, which (in $\frac{1}{2}$ of cases) form an elliptic (cubic) curve in $\mathbb{P}^2$. Our case is the most degenerate case: the curve is $Z^3 = 0$].

Picture of our $\mathbb{P}^2_q$:



[no points here]

Le Bruyn - style classification of ideals

Lemma. Each ideal M has a unique extension to a sheaf $\mathcal{M}$ over $\mathbb{P}^2_q$ which is trivial over the line at ∞.

Usual methods then give : $\boxed{\mathcal{R} = \coprod_{n\geq 0} C_n}$

where

$$C_n = \left\{ \begin{array}{l} \text{pairs of } n\times n \text{ matrices s.t.} \\ [X,Y]+I \text{ has rank } 1 \end{array} \right\} \Big/ \sim$$

cf: $$\mathrm{Hilb}_n(\mathbb{A}^2) = \left\{ \begin{array}{c} \text{pairs of } n\times n \text{ matrices} \\ \text{s.t. } [X,Y]=0 \end{array} \right\} \Big/ \sim$$

Nakajima remark : $\mathrm{Hilb}_n(\mathbb{A}^2)$ is hyperkähler: deforming its complex structure gives C_n. So

$A_0 \rightsquigarrow A$; $\mathcal{R}_0 \rightsquigarrow \mathcal{R}$.

Cannings - Holland solution

$M \subseteq A = \mathbb{C}[x, \partial/\partial x]$ right ideal.

A acts on $\mathbb{C}[x]$; set

$$e(M) = \{ m.f : m \in M, f \in \mathbb{C}[x] \} \subseteq \mathbb{C}[x]$$

$$= M \otimes_A \mathbb{C}[x] \subseteq A \otimes_A \mathbb{C}[x].$$

<u>Theorem</u> e defines a bijection

$$\left\{ \begin{array}{c} \text{ideals } M \text{ s.t.} \\ M \cap \mathbb{C}[x] \neq \{0\} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{primary decomposable} \\ V \subseteq \mathbb{C}[x] \end{array} \right\},$$

<u>Defn</u>: V is primary decomposable if

$$V = \bigcap_{\lambda \in \mathbb{C}} V_\lambda \,,$$ where

(i) $(x-\lambda)^k \mathbb{C}[x] \subseteq V_\lambda$ for some $k(\lambda)$;

(ii) $V_\lambda = \mathbb{C}[x]$ for nearly all λ.

Lemma: every ideal is isomorphic to one that intersects $\mathbb{C}[x]$. So to get $\mathcal{R}$ we take $\left\{ \begin{matrix} \text{such} \\ \text{ideals} \end{matrix} \right\} \Big/ \, p(x)\, I \sim q(x)\, I$.

So the last Theorem $\Longrightarrow$:

Corollary, e induces a bijection

$$\mathcal{R} \xrightarrow{\cong} Gr^{ad}.$$

The *adelic Grassmannian* Gr^{ad} consists of all $W \subset \mathbb{C}(x)$ of the form

$$W = \prod_{\lambda} (x-\lambda)^{-k_\lambda} V$$

where V is p.d., $k_\lambda = \operatorname{codim} V_\lambda$.

Connection between the 2 solutions

Both the spaces C_n and Gr^{ad} occur in the theory of $\int$ble systems.

① (Kazhdan - Kostant - Sternberg ⊗ $\mathbb{C}$, 1977)

C_n $(= \{(X,Y) : \ldots\})$ is a (the?) completed phase space for the n-particle Calogero-Moser system (x^{-2} potential);

the usual phase space is

$T^*(\mathbb{C}^{(n)} \setminus \Delta) \cong C_n' = \{$ pairs with X d'ble $\}$.

② (Krichever 1978) Gr^{ad} is the space of rational solutions of KP.

③ (Airault – McKean – Moser, Chudnovsky, Krichever). "The motion of the poles of KP is governed by CM"; which means ≙:

<u>Theorem</u> There is a bijective map

$$\coprod_n C_n' \longrightarrow \left\{ \begin{array}{l} \text{part of } Gr^{ad} \text{ where poles} \\ \text{of KP solution are distinct} \end{array} \right\}$$

④ (G.W. 1998). This extends to a bijection

$$\coprod_n C_n \xrightarrow{\cong} Gr^{ad} .$$

(commuting with CM and KP flows).

Unfortunately, this map is mysterious except to ∫ble systems fanatics: it sends (X, Y) to the point of Gr^{ad} with Baker function

$$\tilde{\psi}(x, z) = \det\left\{ I - (xI - X)^{-1}(zI - Y)^{-1} \right\}$$

The automorphism group

Let $G = \mathrm{Aut}(A)$.

Theorem (Dixmier 1968) G is generated by the "tame" automorphisms

$$\Phi_p(x) = x - p'(y), \quad \Phi_p(y) = y$$

$$\Psi_q(x) = x, \quad \Psi_q(y) = y + q'(x).$$

There is a natural action of G on $\mathcal{R}$ ($M \mapsto M^\sigma$). We can tranfer this to the other spaces via our various bijections.

<u>Theorem</u> (Berest–W) The action of G on C_n obtained from $\mathbb{R} \xrightarrow{CH} Gr^{ad} \xleftarrow{IS} \coprod_n C_n$ is given by

$$\Phi_p(X, Y) = (X + p'(Y), Y)$$

$$\Psi_q(X, Y) = (X, Y - q'(X)).$$

<u>Theorem</u> (Berest–W). The action of G on C_n obtained from $\mathbb{R} \xrightarrow{LB} \coprod_n C_n$ is the same.

i.e. the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "LB"] \arrow[dr, "CH"'] & \coprod C_n \arrow[d, "IS"] \\ & Gr^{ad} \end{tikzcd}$$

commutes.

<u>Theorem</u> (Berest–W 1999, K. Kouakou 1994)

G acts <u>transitively</u> on each C_n.

Some funny things

Look at the 3 algebras

$$\begin{tikzcd} & \mathcal{A} = \mathbb{C}\langle x, y\rangle \arrow[dl] \arrow[dr] & \\ \mathbb{C}[x,y] = A_0 & & A = \text{Weyl} \end{tikzcd}$$

and their groups of automorphisms preserving $xy - yx$:

$$(*) \qquad \begin{tikzcd} & \mathcal{G} \arrow[dl] \arrow[dr] & \\ G_0 & & G \end{tikzcd}$$

Theorem (Czerniakiewicz, Makar-Limanov)

Both maps in $(*)$ are isomorphisms.

<u>Algebraic groups</u> (Shafarevich 196?, ...)

Let $\sigma \in G$ (or G_0 or $\mathcal{G}$); set

$$d(\sigma) = \max\{\text{degrees of } \sigma(x), \sigma(y);\ \sigma^{-1}(x), \sigma^{-1}(y)\}$$

Then $G = \bigcup_d G^{(d)}$ ← autos of degree $\leq d$

Each $G^{(d)}$ is a f.d. variety and so G becomes an <u>algebraic</u> group. The maps

$$\begin{tikzcd} & \mathcal{G} \arrow[dl] \arrow[dr] & \\ G_0 & & G \end{tikzcd}$$

are algebraic; but they are NOT isos.

<u>For example</u>:

$$\mathrm{Lie}(G_0) \cong A_0^{\{\,\}}/\mathbb{C}$$

$$\mathrm{Lie}(G) \cong A^{[\,]}/\mathbb{C} \quad ;$$

$\mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(G_0)$ is not injective ...

<u>Conclusion</u>. As we deform $A_0 \rightsquigarrow A$:

$$R_0 \rightsquigarrow R$$

$$(\text{i.e. } \mathrm{Hilb}_n(\mathbb{A}^2) \rightsquigarrow C_n)$$

and $G_0 \rightsquigarrow G$;

so the G_0-action on $\mathrm{Hilb}_n(\mathbb{A}^2)$ deforms to our G-action on C_n. BUT:

<u>Theorem</u> (Hazewinkel-Marcus 1982)
The algebraic group G has no f.d. homogeneous spaces.

So the "natural" action of G on R is not algebraic (however, the action of $\mathfrak{g}$ is).

??? ??????? ???