RANDOM WORDS, RANDOM WALKS, GRASSMANNIAN INTEGRALS, and INTEGRABLE LATTICES

by

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MSRI (Berkeley), April 19, 2002

Integrable systems have solutions satisfying Virasoro constraints. Expanding with respect to a parameter leads to:

- topology
- random matrices
- combinatorics
- statistics

Discuss the following three integrals

- $$\int_{U(\ell)} e^{t\,\mathrm{tr}(M+\bar{M})} dM$$

- $$\int_{U(\ell)} s_\mu(M) s_\nu(\bar{M}) e^{t\,\mathrm{tr}(M+\bar{M})} dM$$

- $$\int_{Gr(p,\mathbb{F}^n)} e^{x\,\mathrm{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-(q-p)} dZ$$

  with $\mathbb{F} = \mathbb{C},\ \mathbb{R},\ \mathbb{H}$.

INTEGRAL I

- A determinant of moments

$$
\begin{aligned}
I_\ell &:= \int_{U(\ell)} e^{t\ \mathrm{tr}(M+\bar{M})} dM \\
&= \det\left(\int_{S^1} e^{t(z+z^{-1})} z^{i-j} \frac{dz}{2\pi i z}\right)_{1\le i,j\le \ell} \\
&= \det\left(J_{i-j}(t)\right)_{1\le i,j\le \ell}
\end{aligned}
$$

(Toeplitz matrix)

using the Bessel function expansion

$$
e^{t(z+z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.
$$

• The length of the longest increasing sequence in a random permutation:

$$
\begin{aligned}
I_\ell &:= \int_{U(\ell)} e^{t\,\mathrm{tr}(M+\bar{M})} dM \\
&= \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \frac{\#\{\pi_n \in S_n \mid L(\pi_n) \le \ell\}}{n!} \quad \text{(Gessel)} \\
&= \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} P_{pl}\begin{pmatrix} \text{length of first column of} \\ \text{partition } \lambda \text{ of } n \text{ is } \le \ell \end{pmatrix}
\end{aligned}
$$

by RSK correspondence

for Plancherel measure : $P_{pl}(\lambda \in \mathbb{Y}_n) := \frac{(f^\lambda)^2}{n!}$

$f^\lambda = \#\{\text{standard tableaux of shape } \lambda\}$

Examples:

$$\begin{cases} \text{for } \pi_7 = (\underline{3}, 1, \underline{4}, 2, \underline{6}, \underline{7}, 5) : & L(\pi_7) = 4. \\ \text{for } \pi_5 = (5, \underline{1}, 4, \underline{3}, 2) : & L(\pi_5) = 2. \end{cases}$$

$$\pi_5 \longrightarrow \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \cline{1-1} 5 & \\ \cline{1-1} \end{array} \;, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1} 5 & \\ \cline{1-1} \end{array} \right)$$

= two standard tableaux of same shape

(Robinson-Shensted-Knuth correspondence)

• Painlevé equations and rational inductive equations :

$$\begin{aligned} I_\ell(t) &:= \int_{U(\ell)} e^{t\ \mathrm{tr}(M+\bar{M})} dM \\ &= \exp \int_0^{t^2} \log\left(\frac{t^2}{u}\right) g_\ell(u)du, \quad \text{(Painlevé V)} \\ &= (J_1(t))^\ell \prod_1^{\ell-1} (1-x_i^2)^{\ell-i} \quad \left(\begin{array}{c}\text{polynomial in} \\ \text{the } x_i\text{'s}\end{array}\right) \end{aligned}$$

with $x_i = x_i(t)$ satisfying

$$x_0 = 1, \quad x_1 = -\frac{J_1(t)}{J_0(t)}$$

$$x_{n+1} = \frac{nx_n}{t(x_n^2-1)} - x_{n-1}$$

MacMillan equation

(rational three step relation)

(Tracy-Widom, Adler-PvM, Borodin, Adler-PvM)

- $$\begin{cases} g'' - \dfrac{g'^2}{2}\left(\dfrac{1}{g-1} + \dfrac{1}{g}\right) + \dfrac{g'}{u} + \dfrac{2}{u}g(g-1) - \\ \qquad\qquad \dfrac{\ell^2}{2u^2}\dfrac{g-1}{g} = 0 \\ \qquad\qquad \textbf{(Painlevé V)} \\ \text{with } \; g_\ell(u) = 1 - \dfrac{u^\ell}{(\ell!)^2} + O(u^{\ell+1}), \\ \qquad\qquad \text{near } u = 0. \end{cases}$$

These facts follow from a basic observation: insert time parameters $(t_1, t_2, \ldots)$ and $(s_1, s_2, \ldots)$ (with $x_0(t,s) = y_0(t,s) = 1$) in the ratios of integrals:

$$x_n(t,s)$$

$$:= \ (-1)^n \frac{\int_{U(n)} \det(M) e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM}{\int_{U(n)} e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM}$$

$$y_n(t,s)$$

$$:= \ (-1)^n \frac{\int_{U(n)} \det(\bar{M}) e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM}{\int_{U(n)} e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM}.$$

satisfy

(i) The Toeplitz lattice

(ii) Virasoro constraints

(Adler-PvM)

(i) Toeplitz lattice: The matrices $L_1(t)$ and $L_2(t)$

$$L_1 := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}.$$

satisfy the equations ($i = 1, 2$ and $n \geq 1$)

$$\frac{\partial \hat{L}_i}{\partial t_n} = \left[\left(\hat{L}_1^n\right)_+, \hat{L}_i\right] \quad \text{and} \quad \frac{\partial \hat{L}_i}{\partial s_n} = \left[\left(\hat{L}_2^n\right)_-, \hat{L}_i\right]$$

(Two-Toda Lattice)

where

$$\hat{L}_1 := hL_1h^{-1} \quad \text{and} \quad \hat{L}_2 := L_2,$$

Equivalently, the x_n and y_n satisfy the following integrable Hamiltonian system

$$\frac{\partial x_n}{\partial \left\{ \begin{smallmatrix} t_i \\ s_i \end{smallmatrix} \right\}} = (1 - x_n y_n) \frac{\partial H_i^{\binom{1}{2}}}{\partial y_n} \quad (1)$$

$$\frac{\partial y_n}{\partial \left\{ \begin{smallmatrix} t_i \\ s_i \end{smallmatrix} \right\}} = (1 - x_n y_n) \frac{\partial H_i^{\binom{1}{2}}}{\partial x_n} \quad (2)$$

(Toeplitz lattice)

- initial condition $x_n(0,0) = y_n(0,0) = 0$ for $n \geq 1$
- boundary condition $x_0(t,s) = y_0(t,s) = 1$.
- $H_i^{(k)} = -\frac{1}{i} \text{Tr } L_k^i, \quad i = 1, 2, 3, ..., \quad k = 1, 2$

(ii) Virasoro constraints:

The integrals

$$\tau_n(t,s) = \int_{U(n)} (\det M)^{\gamma} e^{\sum_1^{\infty} \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

satisfy an SL(2,$\mathbb{Z}$)-algebra of Virasoro constraints:

$$\mathbb{V}^{\gamma}_{k,n} \tau_n(t,s) = 0, \quad \text{for } \left\{ \begin{array}{l} k = -1, 0, 1 \\ n = 0, 1, 2, 3, \dots \end{array} \right\}$$

with the vector Virasoro constraint

$$\begin{aligned} \mathbb{V}^{\gamma}_{k} &:= (\mathbb{V}^{\gamma}_{k,n})_{n \geq 0} \\ &:= \mathbb{J}^{(2)}_{k}(t) - \mathbb{J}^{(2)}_{-k}(-s) \\ &\qquad -(k-\gamma)\Big(\theta \mathbb{J}^{(1)}_{k}(t) + (1-\theta)\mathbb{J}^{(1)}_{-k}(-s)\Big). \end{aligned}$$

2d-Toda relation: (example)

$$(*)\quad \frac{\partial^2}{\partial s_2 \partial t_1}\log\tau_n = -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} \cdot \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n.$$

Three Virasoro generators:

$$\mathbb{V}^{\gamma}_{-1,n}\tau^{\gamma}_n = \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + nt_1 + (n-\gamma)\frac{\partial}{\partial s_1}\right)\tau^{\gamma}_n = 0$$

$$\mathbb{V}^{\gamma}_{0,n}\tau^{\gamma}_n = \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right)\tau^{\gamma}_n + \gamma n\tau^{\gamma}_n = 0$$

$$\mathbb{V}^{\gamma}_{1,n}\tau^{\gamma}_n = \left(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} + ns_1 + (n+\gamma)\frac{\partial}{\partial t_1}\right)\tau^{\gamma}_n = 0.$$

Used in (*) to eliminate derivatives wrt higher s_i, t_i for $i \geq 2$, like $\frac{\partial^2}{\partial s_2\partial t_1}\log\tau_n$.

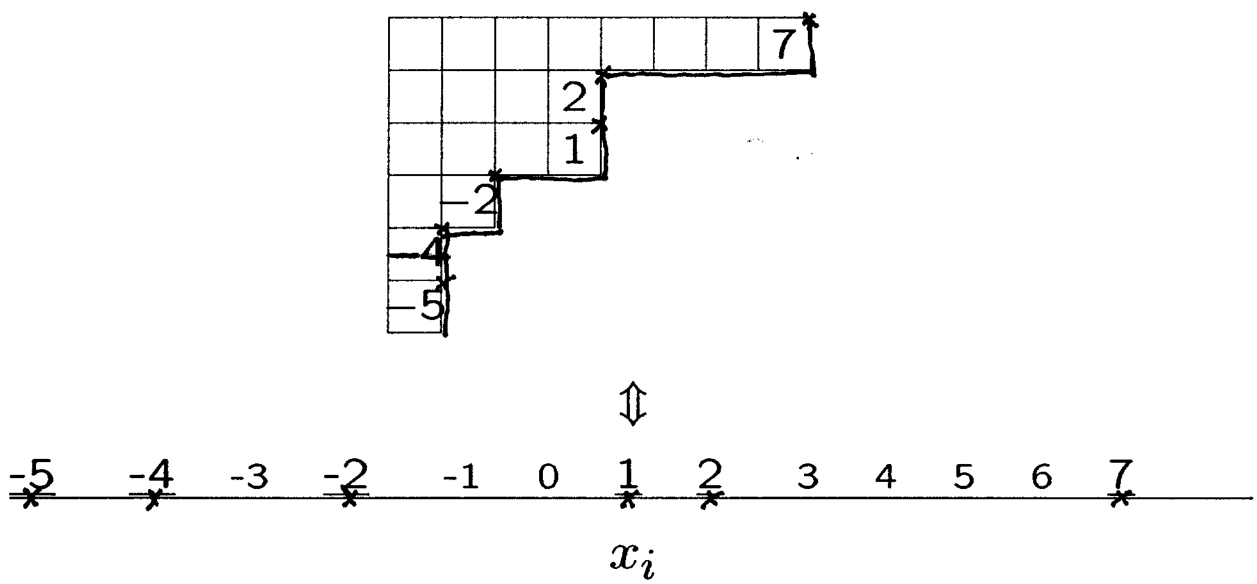
INTEGRAL 2

$$\int_{U(n)} s_\mu(M) s_\nu(\bar{M}) e^{z \operatorname{tr}(M+\bar{M})} dM$$

Define

$$x_i := \mu_i - i \quad \text{and} \quad y_i := \nu_i - i$$

<u>*Example:*</u> $\mu = (8, 4, 4, 2, 1, 1)$



$$\int_{U(n)} s_\mu(M) s_\nu(\bar{M}) e^{z\,\mathrm{tr}(M+\bar{M})} dM$$

$$= \det\left(J_{y_i - x_j}(z)\right)_{1 \le i,j \le n}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \# \left\{ \begin{array}{l} n \text{ non-intersecting walkers} \\ \text{going in } k \text{ steps} \\ \text{from } x_1 < x_2 < \ldots < x_n \\ \text{to } y_1 < y_2 < \ldots < y_n \end{array} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k_L! k_R!} \sum_{\substack{\lambda \text{ such that } \lambda_1 \le n \\ |\lambda \backslash \mu| = k_L,\ |\lambda \backslash \nu| = k_R}} f^{\lambda \backslash \mu} f^{\lambda \backslash \nu}$$

where

- $e^{z(t+t^{-1})} = \sum_{-\infty}^{\infty} t^n J_n(z)$

- $k_{\left\{ \begin{smallmatrix} L \\ R \end{smallmatrix} \right\}} = \frac{1}{2}(k \pm (y_1 - x_1))$

- $f^{\lambda \backslash \mu} = \# \left\{ \begin{array}{l} \text{standard skew-tableaux filled} \\ \text{with numbers } 1, \ldots, |\lambda \backslash \mu| \end{array} \right\}$

(Fischer, Gessel, Forrester, Grabiner, Baik, Adler-Shiota-PvM)

INTEGRAL 3.

$$\int_{Gr(p,\mathbb{F}^n)} e^{x\,\mathrm{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-(q-p)} dZ$$

with $\mathbb{F} = \mathbb{C},\ \mathbb{R},\ \mathbb{H}.$ (Adler-PvM)

(dZ=left-invariant measure on the symmetric space $Gr(p, \mathbb{F}^n)$ in appropriate coordinates)

Relates to:

- random words

- another probability measure on Young diagrams ($\neq$ Plancherel measure)

- testing statistical independence of Gaussian populations

Upon putting an appropriate $(t_1, t_2, \dots)$ dependence, this integral becomes a solution of

$$\begin{cases} \text{Toda lattice} & (\mathbb{F} = \mathbb{C}) \\ \text{Pfaff lattice} & (\mathbb{F} = \mathbb{R} \text{ or } \mathbb{H}) \end{cases}$$

This is joint work with Mark Adler

Uniform probability:

$$P^{\ell,p}(\pi) = \frac{1}{p^\ell}.$$

on

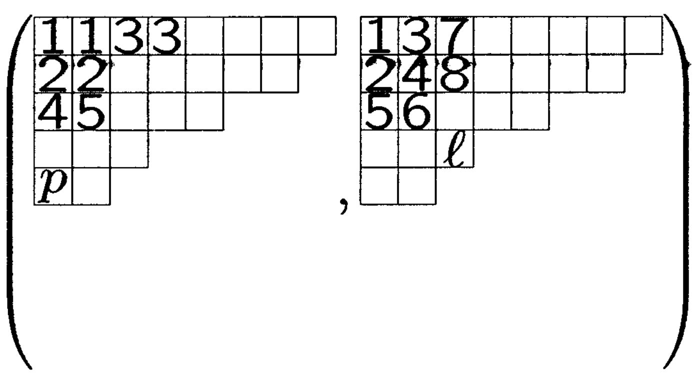
$$S_\ell^p := \left\{ \begin{array}{l} \text{words } \pi \text{ of length } \ell, \text{ from} \\ \text{an alphabet } \{1, ..., p\} \end{array} \right\}$$

The **RSK correspondence**:

$$\text{word} \in S_\ell^p$$

$$\Updownarrow$$

$$\Big(\text{semi-standard tableau } P, \text{ standard tableau } Q\Big)$$



where P and Q have the form

$$\lambda \in \mathbb{Y}_\ell = \{\text{partitions } \lambda \in \mathbb{Y} \text{ of weight } |\lambda| = \ell\}.$$

Induces a **probability on partitions**

$\lambda \in \mathbb{Y}_\ell = \{$partitions $\lambda \in \mathbb{Y}$ of weight $|\lambda| = \ell\}$,
given by

$$P^{\ell,p}(\lambda) = \frac{f^\lambda \; s_\lambda(1^p)}{p^\ell}, \quad |\lambda| = \ell,$$

where

- s_λ Schur polynomials associated with λ

- $1^p = (\overbrace{1, \ldots, 1}^{p}, 0, 0, \ldots)$

- $f^\lambda = \#\{$standard tableaux of shape $\lambda\}$

- $s_\lambda(1^p) = \#\left\{\begin{array}{l}\text{semi-standard tableaux} \\ \text{of shape } \lambda\text{, containing} \\ \text{the integers } (1, \ldots, p)\end{array}\right\}$

Based on Cauchy's identity:

$$\prod_{i,j \geq 1} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathbb{Y}} s_\lambda(x) s_\lambda(y).$$

Subsequence σ of the word π is k-**increasing:**

$$\sigma = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k,$$

$$\sigma_i = \text{increasing subsequences of } \pi.$$

By the RSK correspondence: (Greene)

$$d_1(\pi) = \left\{ \begin{array}{l} \text{maximal length of} \\ \text{a strictly decreasing} \\ \text{subsequence of } \pi \end{array} \right\} = \lambda_1^\top = \left\{ \begin{array}{c} \text{length of} \\ \text{first column} \\ \text{of } \lambda_1 \end{array} \right\}$$

$$i_k(\pi) = \left\{ \begin{array}{l} \text{maximal length of} \\ \text{the \textit{disjoint union}} \\ \text{of } k\text{-increasing} \\ \text{subsequences of } \pi \end{array} \right\} = \lambda_1 + \ldots + \lambda_k$$

Example:

$$\pi = (4, \underline{2}, \underline{3}, 6, \underline{5}, 1, \underline{7}) \longrightarrow (2,3,5,7) \qquad i_1(\pi) = 4$$

$$\pi = (\underline{\underline{4}}, \underline{2}, \underline{3}, \underline{\underline{6}}, \underline{5}, 1, \underline{7}) \longrightarrow (2,3,5,7) \cup (4,6)$$

$$i_2(\pi) = 4 + 2 = 6$$

Consider a word such that:

$$d_1(\pi) = p$$

$$\Updownarrow$$

$$\pi = (\dots, p, \dots, p-1, \dots, p-2, \dots, 2, \dots, 1, \dots)$$

(word of length ℓ)

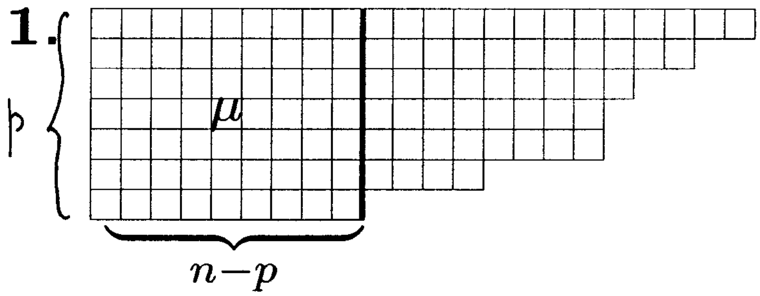
Then

$$\begin{aligned} i_p(\pi) &= \ell = \text{ trivial} \\ i_{p-1}(\pi) &= \text{ non-trivial} \end{aligned}$$

↗

This is precisely what the probability on the next slide will measure

FACT 1.



n is a ≥0 integer

$$P^{\ell,p}\left(\pi \in S_\ell^p \;\middle|\; \begin{array}{c} d_1(\pi) = p \text{ and} \\ i_{p-1}(\pi) \le \ell - (n-p) \end{array}\right)$$

$$= \frac{\ell!}{p^\ell} \prod_1^p \frac{(p-i)!}{(n-i)!} \sum_{\substack{\kappa \in \mathbb{Y}_{\ell - p(n-p)} \\ \kappa_1^\top \le p}} \frac{1}{(h^\kappa)^2} \frac{((p)_\kappa)^2}{(n)_\kappa}$$

$$= P^{\ell,p}(\lambda \supseteq \text{ fixed } \mu)$$

μ rectangular

<u>Notation</u>: $(a)_\lambda := \prod_i (a+1-i)_{\lambda_i}$

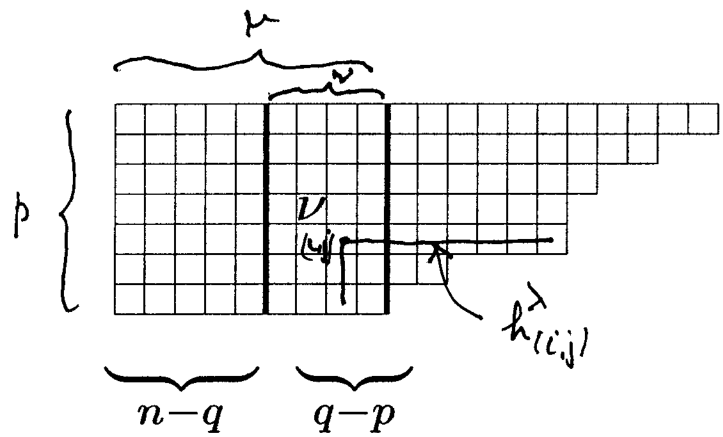
where $(x)_n := x(x+1)\dots(x+n-1)$, $x_0 = 1$

$h^\kappa = \prod(\text{"hook length" of all } (i,j) \in \kappa)$

Hypergeometric function in many variables:

$${}_2F_1^{(1)}(p,q;n;x) := \sum_{\kappa \in \mathbb{Y}} \frac{1}{(h^\kappa)^2} \frac{(p)_\kappa (q)_\kappa}{(n)_\kappa}\; h^\kappa s_\kappa(x).$$

FACT 2.



For $p \leq q < n$: (fixed rectangular μ)

$$E^{\ell,p}\left(I_{\{\lambda \supseteq \mu\}}(\lambda) \prod_{\substack{(i,j)\in\lambda \\ n-q<j\leq n-p}} h^{\lambda}_{(i,j)} \right)$$

$$= \frac{\ell!}{p^{\ell}} \prod_{1}^{p} \frac{(q-i)!}{(n-i)!} \sum_{\substack{\kappa \in \mathbb{Y}_{\ell-p(n-p)} \\ \kappa_1^{\top} \leq p}} \frac{1}{(h^{\kappa})^2} \frac{(p)_{\kappa}(q)_{\kappa}}{(n)_{\kappa}}.$$

Reminder:

$$P^{\ell,p}(\lambda) = \frac{f^{\lambda}\ s_{\lambda}(1^p)}{p^{\ell}}, \quad |\lambda| = \ell,$$

FACT 3. (Adler-PvM) For $p \le q \le n/2$, consider the **generating function**

$$cx^{-p(n-p)} \sum_{\ell \ge p(n-p)} \frac{(px)^{\ell}}{\ell!} E^{\ell,p} \left(I_{\{\lambda \supseteq \mu\}}(\lambda) \prod_{\substack{(i,j)\in\lambda \\ n-q<j\le n-p}} h^{\lambda}_{(i,j)} \right)$$

$$= \quad {}_2F_1^{(1)}(p,q;n;y)\Big|_{\sum_\ell y_\ell^i = \delta_{1i}x}$$

$$= \tilde{c} \int_{Gr(p,\mathbb{C}^n)} e^{x\,\mathrm{Tr}(I+Z^{\dagger}Z)^{-1}} \det(Z^{\dagger}Z)^{-(q-p)} dZ$$

$$= \exp \int_0^x \frac{u(y) - p(n-p) + py}{y} dy$$

where $u(x)$ is the unique solution of the equation:

$$\begin{cases} x^2u''' + xu'' + 6xu'^2 - 4uu' + 4Qu' - 2Q'u + 2R = 0 \\ \qquad\qquad\qquad\qquad \textbf{(Painlevé V)} \\ \text{with} \\ u(x) = p(n-p) - \dfrac{p(n-q)}{n}x + \ldots + a_{n+1}x^{n+1} \\ \qquad\qquad +O(x^{n+1}) + \ldots, \text{ near } x = 0. \end{cases}$$

where

$$
\begin{aligned}
4Q &= -x^2 + 2(n + 2(p - q))x - (n - 2p)^2 \\
2R &= p(p - q)(x + n - 2p).
\end{aligned}
$$

To explain the Grasmannian integral,
see next three slides.

Integration on Grassmannians

Consider $(p \leq n/2)$

$$Gr(p, \mathbb{C}^n) = \frac{U(n)}{U(p) \times U(n-p)} =: G/K$$

$$\begin{matrix} p \updownarrow \\ n-p \updownarrow \end{matrix} \overset{\quad\overset{p}{\longleftrightarrow} \quad \overset{n-p}{\longleftrightarrow}}{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}} \in G$$

$$\begin{pmatrix} B_{11} & O \\ O & B_{22} \end{pmatrix} \in K.$$

$$\begin{aligned} \text{Affine } Gr(p, \mathbb{F}^n) &= \left\{ \text{span } A \begin{pmatrix} I_p \\ O \end{pmatrix} \Big\updownarrow n, \text{ where } A \in G \right\} \\ &= \left\{ \text{span } \begin{pmatrix} A_{11} \\ A_{21}^{-1} \end{pmatrix}, \text{ where } A \in G \right\} \\ &= \left\{ \text{span } \begin{pmatrix} I_p \\ A_{21} A_{11}^{-1} \end{pmatrix} \;\Big|\; A \in G \right\} \end{aligned}$$

and

$$K\backslash G/K$$

$$= \left\{ \text{span } KA \begin{pmatrix} I_p \\ O \end{pmatrix}, \text{ où } A \in G \right\}$$

$$= \left\{ \text{span } \begin{pmatrix} I_p \\ B_{22}A_{21}A_{11}^{-1}B_{11}^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} A \in G, \\ \text{all } B \in K \end{matrix} \right\}$$

$$= \left\{ \text{span } \begin{pmatrix} I_p \\ Z \end{pmatrix} \;\middle|\; Z = \begin{pmatrix} -\tan\lambda_1 & & O \\ & \ddots & \\ O & & -\tan\lambda_p \\ \hline & O & \end{pmatrix} \right.$$

Setting

$$z_i \;:=\; \frac{1}{1+\tan^2\lambda_i} = \cos^2\lambda_i, \quad 1 \le i \le p$$

where $0 \le z_i \le 1$, is Haar measure on $Gr(p, \mathbb{C}^n)$ of the following shape, for $\beta = 1$,

$$d\mu(Z) = |\Delta_p(z)|^{2\beta} \prod_1^p (1 - z_i)^{\beta(n-2p)}((1 - z_i)z_i)^{\beta-1} dz_i$$

(Weyl integration formula)

Thus the integral on $Gr(p, \mathbb{C}^n)$ has the form (for $\beta = 1$):

$$\int_{Gr(p,\mathbb{C}^n)} e^{x\,\mathrm{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)} d\mu(Z)$$

$$= c \int_{[0,1]^p} e^{x \sum_1^p z_i} |\Delta_p(z)|^{2\beta}$$

$$\prod_1^p z_i^{\beta(q-p+1)-1} (1-z_i)^{\beta(n-q-p+1)-1} dz_i.$$

Statistical independence of Gaussian populations

$$(X_1, ..., X_p) \text{ et } (Y_1, ..., Y_q), \qquad (p \leq q)$$

are $p+q$ normal random variables, centered at zero, whose covariance matrice is given by

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^{\top} & \Sigma_{22} \end{pmatrix}.$$

Non-singular linear transformations on the X's and Y's lead to the canonical form (Hotelling)

$$\Sigma_{canonical} = \begin{pmatrix} I & P & 0 \\ P & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad P = \text{diag}(\rho_1, ..., \rho_p),$$

où $\rho_1 \geq \rho_2 \geq ... \geq \rho_p$,

$$\begin{aligned} \rho_i &= \textit{canonical correlation coefficients} \\ &= \textit{roots of } \det(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{\top} - \rho^2\Sigma_{11}) = 0. \end{aligned}$$

The $n \geq p+q$ independent samples of $(X_1, ..., X_p)$ and $(Y_1, ..., Y_q)$:

$$(x_{11}, ..., x_{1p}, y_{11}, ..., y_{1q})^{\top}, \quad ..., (x_{n1}, ..., x_{np}, y_{n1}, ..., y_{nq})^{\top}$$

form a matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} = (p+q) \times n \text{ matrix.}$$

The maximum likelihood estimators r_i of ρ_i satisfy the equation

$$\det(S_{12}S_{22}^{-1}S_{12}^{\top} - r^2 S_{11}) = 0,$$

where

$$S = \begin{pmatrix} xx^{\top} & xy^{\top} \\ yx^{\top} & yy^{\top} \end{pmatrix} = \left\{ \begin{array}{l} \textit{sample} \text{ covariance} \\ \text{matrix} \end{array} \right. .$$

The $z_i := r_i^2 = \cos^2\theta_i$ $(\frac{1}{2}\pi > \theta_1 > \ldots > \theta_p > 0)$ have the density (James and Constantine)

$$c_{p,q,n}|\Delta(z)|^{2\beta} \prod_{i=1}^{p} (1-z_i)^{\beta(n-p-q+1)-1} z_i^{\beta(q-p+1)-1} dz_i$$

for $\beta = \frac{1}{2}$.

Its "Laplace transform" is given by

$$c \int_{[0,1]^p} e^{x \sum_1^p z_i} |\Delta_p(z)|^{2\beta} \prod_1^p z_i^{\beta(q-p+1)-1}(1-z_i)^{\beta(n-q-p+1)-1} dz_i$$

$$= \int_{Gr(p,\mathbb{R}^n)} e^{x \operatorname{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)} d\mu(Z)$$

for $\beta = 1/2$.

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