(I)

OPERATOR ALGEBRAIC ANALYSIS OF SOLVABLE LATTICE MODELS

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(1) Physics Background

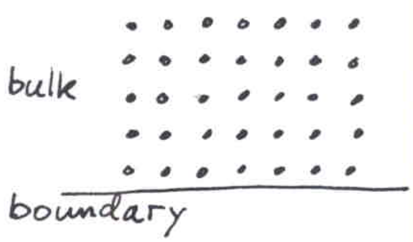
In 1980s, two ideas introduced to understand solvable lattice models (SLM)

- R. Baxter — Corner Transfer Matrix (CTM)
- J. Cardy — Boundary conditions in Conformal Field Theory (BCFT)

Want to understand: why $\boxed{\text{"CFT is scaling limit at criticality of SLM."}}$

known: for Ising model (McCoy, Wu, et al & Kyoto school)

proposed method: use operator algebras (von Neumann & C^* algebras)



- lattice model – encoded in a braid group repn  with spectral parameter z and deformation parameter $q \in (0,1)$.
- boundary data given by reflection equation of Cherednik & Sklyanin . Satisfies pure braid type relns and depends also on z and q.

Aim is to compute correlation functions which describe a massive integrable field theory in the sense of F. Smirnov. At criticality, $q \to 1$, expect to get a CFT which is a theory living on a Riemann surface in the sense of Tsuchiya and Segal. The boundary theory should live on a Riemann surface with a $\mathbb{Z}_2$ symmetry $z \mapsto \bar{z}$. The boundary corr. to $z = \bar{z}$, rest is the bulk. A major outstanding problem is to give the operator formalism in this case and describe the boundary data (labels for boundary fields). The idea of "quantum subgroup" that has emerged from the theory of subfactors seems to provide a step in this direction.

It is worth adding that BCFT has had 2 successes. Affleck used it to explain the Kondo effect for electrons moving in a magnetic field, so in agreement with experiment. And BCFT for the $q=0$ Potts model has led to important work in prob. theory on percolation by Smirnov and Lawler-Schramm-Werner. Smirnov proved the Cardy-Carleson formula for percolation in a triangle and LSW showed that could make sense of scaling limit in terms of self-erasing random walk.

II

② C^* dynamics formulation of Baxter's CTM Hamiltonian

(Periodicity and Uniqueness)

Normally in a lattice model (in this case the 1984 Andrew-Baxter-Forrester model), the partition function or 1 pt function is defined by taking the average over a large square. For each plaquette or small square one labels the states of the vertices by a spin in $\{0, \frac{1}{2}, \dots, \frac{\ell}{2}\}$

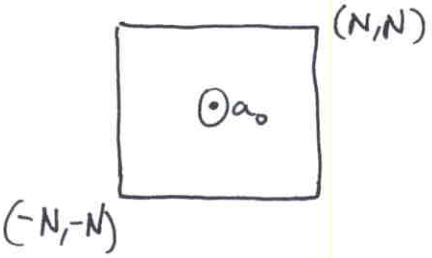
$\{0, \frac{1}{2}, \dots, \frac{\ell}{2}\} \ni a$ $\quad$ b

d $\quad$ c

The Boltzmann weight $W_{z,q}\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ attaches a probability to each such plaquette

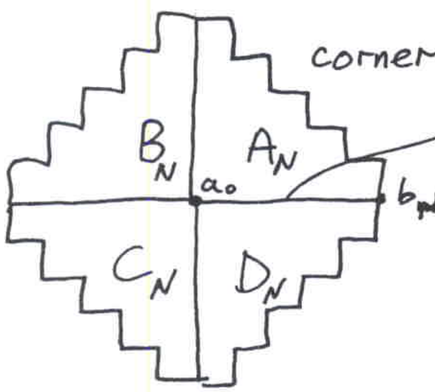
It is $\neq 0$ only if neighbouring vertices satisfy $|x-y| = \frac{1}{2}$.

The 1 pt function is def. as

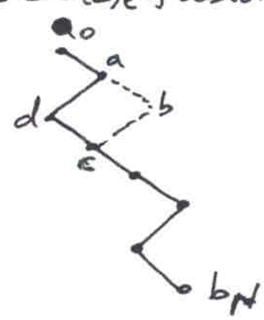


$$P_{a_0} = \lim_{N\to\infty} \frac{\sum_{\text{states}, 0=a_0} \prod W}{\sum_{\text{all states}} \prod W}$$

Baxter's CTM method rotates this large square thro' 45°.



For this segment have states given by paths moving on $\{0, \frac{1}{2}, \dots, \frac{\ell}{2}\}$ by steps of $\frac{1}{2}$ (the $SU(2)_\ell$ fusion rule!)



$W_{z,q}\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ act on space of paths at the n and $(n+1)$ segments – it gives the matrix coefficient. (At stage 2 and 3 indicated above) Making n vary, get braid group generators with coeffs z, q. Taking matrix units e_{IJ} indexed by paths of length N, get a local algebra of observables $\mathfrak{A}_N$ with $\mathfrak{A}_N \subset \mathfrak{A}_{N+1}$. The union of these finite dimensional C^* algebras gives $\mathfrak{A} = \bigcup \mathfrak{A}_N$ algebra of all local observables. Its norm completion for the operator norm is $\overline{\mathfrak{A}}$, the C^* algebra of quasilocal observables. Thus Boltzmann wts $\Rightarrow$ braiding matrices $R_{n,n+1}(z,q) \in \mathfrak{A}_{n+1}$

III The CTM $A_N(z,q)$ correspond to the half twist on N strings (with parameters). The corr contribution to the sum over states is $\mathrm{Tr}(A_N B_N C_N D_N)$.

In 1980 Baxter made following predictions based on computer experiments (see the 2 reprinted pages from his J Stat Phys 15 article):

(a) $A_N(q,z)$ (suitably normalised) $\longrightarrow z^{H_{CTM}(q)}$ as $N \to \infty$,
where $H_{CTM}(q) = \sum_{n\geq 1} n\, r_{n,n+1}(q)$ (CTM Hamiltonian)
and $r(q) = \frac{d}{dz} R(q,z)\big|_{z=1}$.

(b) $H_{CTM}(q)$ has spectrum in $\{0,1,2,\ldots\}$ of finite mult. indep of q

To make sense of (b) (have nothing to say about (a) at present) note that $\delta^q(a) = i[H,a]$ defines a derivation of $\mathcal{A}$. Baxter suggested in article above using Banach + C^* algebras to make sense of (b) analytically.

Theorem 1 (Periodicity) δ^q exponentiates to a periodic 1-parameter automorphism group α_t^q of $\overline{\mathcal{A}}$, $\alpha_t^q(a) = e^{t\delta^q}\cdot a$

Although we have used the exponential, for $q \in (0,1)$, operators $a \in \mathcal{A}$ will only be C^∞ vectors for α_t^q not analytic (see Bratteli & Robinson Vol I).

Now following Glimm & Jaffe, look for ground states of α_t^q. Easiest to describe through associated positive energy (p.e) representations.

Look for $\begin{cases} \pi: \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}) & \text{*-repn on H-Space } \mathcal{H} \\ U_t \text{ periodic unitary gp on } \mathcal{H} & (\text{action of } \mathbb{R}/\mathbb{Z}) \end{cases}$

such that
- covariant $\quad U_t \pi(a) U_t^{-1} = \pi(\alpha_t(a))$
- p.e. $\quad \mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}(n)$, $U_t = e^{2\pi i n t}$ on $\mathcal{H}(n)$ f.d.

In example $\dim \mathcal{H}(0) = 1$ and if Ω = unit vector in $\mathcal{H}(0)$ the p.e repn is entirely determined by the ground state $\varphi(a) = (\pi(a)\Omega, \Omega)$. For $q=0$, the C^* dynamical system is classical – the $r_{n,n+1}(0)$'s commute. So $H_{CTM}(0)$ can be simult. diagonalised – it acts by an energy function on paths. In this case crystal combinatorics $\Longrightarrow$ finitely many irred p.e. repns at $q=0$ indexed by the infinite paths of ABF.

III$_1$

J. Stat Phys 15

Taking logarithms in (50a), applying the inverse orthogonal transformation P^{-1} to obtain $\ln A$, and assuming [as is implied by conjecture (ii)] that P depends on q and x, but not z, we obtain (ignoring terms negligible for n large)

$$\ln A = \tfrac{1}{2}n(n+1)\ln\kappa + [\ln(xz)]W \tag{90}$$

where the operator W is independent of z. Taking the limit $z \to x^{-1}$ (i.e., $v \to -\eta$), b and c become small, and we can use our perturbation expansion result (58)–(60) to evaluate W. The result obtained is that given in conjectures (ii) and (iii).

Thus conjecture (iv), together with the independence of P on z, implies the three previous conjectures.

8. CONCLUSIONS

If the above conjectures are correct, then the corner transfer matrices may well provide a useful tool for handling exactly soluble models in the thermodynamic limit, since then they have an extremely simple diagonal form.

To do this properly it would be necessary to set up appropriate Banach algebras to handle the resulting infinite-dimensional operators. Presumably these algebras would be related to the C^* algebra.[13] Here we have avoided this problem.

In all previous work q, x, and z have been rather buried in the mathematics, though it has been remarked that they are "natural" parameters to use.[4] However, now we can give x and z a direct physical definition, for from (50) and (89)

$$x = \Lambda_{A,1}\Lambda_{B,1}/\Lambda_{A,0}\Lambda_{B,0} \tag{91}$$

$$z = \Lambda_{A,1}\Lambda_{B,0}/\Lambda_{A,0}\Lambda_{B,1} \tag{92}$$

where $\Lambda_{A,0}$ and $\Lambda_{A,1}$ ($\Lambda_{B,0}$ and $\Lambda_{B,1}$) are the largest and next largest, respectively, eigenvalues of the corner transfer matrix A (B).

All the conjectures of course agree with the second-order perturbation expansions obtained. This evidence may not seem very convincing to the reader, but if the calculations are worked through in detail the number of cancellations needed, and that occur, for this to happen becomes quite impressive.

More significantly, we can in fact prove conjectures (i) and (ii) in general, and all four conjectures for the Ising model case, when $ab = cd$, subject to a nonrigorous treatment of the limit of n large. It is intended to publish these results shortly.

III₂

It is interesting to note that our conjecture (iv) implies that any operator of the form (58) has the diagonal form (in the arrow representation)

$$\frac{1}{2}\sum_{j=1}^{n-1} j(\lambda s_j s_{j+1} + \mu) \tag{93}$$

where λ and μ are scalars and we have ignored negligible contributions of type ϕ_n, R_n. Thus all such operators are related to one another, within additive and multiplicative scalars, by orthogonal similarity transformations. These transformations must form a group, and it may be that a way to prove conjecture (iv) would be to examine this group.

One of the delightful properties of the Ising model is that *all* the 2^n by 2^n matrices that occur form a group, and that the members of this group can be represented by $2n$ by $2n$ matrices.[14] Unfortunately, no such property is known for the general six- and eight-vertex models. A rather ambitious hope is that by examining the CTM's we may stumble on such a group, that the solution of the models may thereby be simplified, and even that it may then be possible to solve new models, such as a staggered eight-vertex model.

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(IV)

<u>Theorem 2</u> (Uniqueness) Every ground state (so p.e. repn) admits at most one analytic continuation to a ground state for $q \in (-1,1)$.

Thus given $\varphi_0(a) = (\pi_0(a)\Omega_0, \Omega_0)$ we have only one choice $\varphi_q(a) = (\pi_q(a)\Omega_q, \Omega_q)$ s.t. $q \mapsto \varphi_q(a)$ is analytic for $a \in \mathcal{A}$.
Return to <u>existence theorem</u> at end of talk.

(3) <u>Boundary Conditions for Critical Lattice Models</u> (subfactor method)

A <u>von Neumann algebra</u> M is just the centraliser algebra or commutant of a unitary group. A simple v.N. alg is called a <u>factor</u> and is classified by its <u>modules</u>

Galois theory of inclusions of v.N. algs is most easily encoded in <u>bimodules</u> over same alg for convenience

$$_M X_M \qquad M, M^{op} \text{ bimodule}$$

The key notion is <u>Connes fusion</u> or relative $\otimes$ product. To define this, need <u>vacuum bimodule</u>. This is a Hilbert Space H_0 with a vacuum vector Ω and a conj. linear isometry J st. $J^2 = 1$, $J\Omega = \Omega$ and $JMJ = M'$. (Such an H_0 is ess. unique.) To make H_0 a bimodule use $x \mapsto Jx^*J$ to give action of M^{op}.

<u>Fusion of bimodules</u> $_M X_M$ (for simplicity – else replace M by $M \oplus N$ to get general case)
Always have a vacuum or trivial bimodule $H_0 = L^2(M)$ for M and M^{op}. There is a vacuum vector Ω (= trace vector in case M is a type II_1 factor) and J is a conj. linear isometry fixing Ω s.t. $JMJ = M'$. Thus $x \mapsto Jx^*J$ gives $M^{op} \xrightarrow{\sim} M'$.
Given bimodules X, Y set

$$\mathcal{X} = \mathrm{Hom}_{M^{op}}(H_0, X) \ni x \mapsto \xi = x\Omega$$

$$\mathcal{Y} = \mathrm{Hom}_M(H_0, Y) \ni y \mapsto \eta = y\Omega$$

Thus replace <u>states</u> ξ, η by the <u>fields</u> x, y that create them from the vacuum. (This is a generalisation of the Gelfand–Naimark–Segal construction.) Define an inner product on the alg $\otimes$ product $\mathcal{X} \otimes \mathcal{Y}$ by the <u>four-point function formula</u>

$$\|x \otimes y\|^2 = (x^*x\, y^*y\, \Omega, \Omega)$$

The Hilbert space completion $X \boxtimes Y$ is the <u>fusion</u> of X and Y. It is naturally associative and $H_0 \boxtimes X = X = X \boxtimes H_0$, so can think of H_0 as "trivial" repn.

(V)

Crash Course in Subfactors

If X is a bimodule, so is $\overline{X}$ (the conjugate Hilbert space). Look at irreducible bimodules X s.t. $X \boxtimes \overline{X}$ and $\overline{X} \boxtimes X \supset H_0$.

In $\underbrace{X \boxtimes \overline{X}}_{e_1} \!\!\underbrace{\boxtimes X}_{e_2}$ take projn e_1 onto $H_0 \boxtimes X$ and projn e_2 onto $X \boxtimes H_0$. Get the Jones relations

$$e_1 e_2 e_1 = \tau e_1 \quad \text{and} \quad e_2 e_1 e_2 = \tau e_2$$

where $\tau^{-1/2}$ = quantum dimension of X and τ^{-1} = Jones index of the inclusion $\pi_X(M) \subset \pi_X(M^{op})'$. Clearly an irreducible bimodule is equivalent to such an inclusion or subfactor. Finite depth means that the irred bimodule and its conj generate a finite system of irreds closed under fusion and taking conjugates. Popa showed this "unitary tensor category" classifies the bimodule or subfactor. There are many possible axiomatisations of such systems. Ocneanu uses the 6j symbols of the corr $\otimes$ category, while Popa uses λ-lattices. The main features in all approaches are UNITARITY & POSITIVITY which allow passage to v.N. algs.

Classical example Let G be a finite group and $M = \left(\bigcup_{n \geq 1} \operatorname{End}_G V^{\otimes n}\right)''$ a type II_1 factor. Have the "invariant theory" inclusion $M \hookrightarrow M$, $x \mapsto 1 \otimes x$. The bimodules for this inclusion are given by (the H. sp. completions of) $\bigcup_{n \geq 1} \operatorname{Hom}_G(V^{\otimes n}, W \otimes V^{\otimes n})$ labelled by irreps W of G. Their fusion reproduces Rep G.

If $H \leq G$, another inclusion given by $M \subset N = \left(\bigcup_{n \geq 1} \operatorname{End}_H V^{\otimes n}\right)''$. In this case may regard $L^2(N)$ (H. Sp. completion of N) as an algebra in the category of M-bimodules (see my talk of Dec 2000 in MSRI). Translating to Rep(G), find $\ell^\infty(G/H) = A$. It is thus an Abelian ergodic algebra. This gives hint as to how to find "subgroup" phenomenon for more general subfactors or unitary $\otimes$ cats. Note that $\ell^\infty(G/H)$ is an "imprimitivity alg" in sense of Mackey — (A, G)-modules corr. exactly to H-modules (by induction).

Jones "lattice model" construction (GHJ)

Building $SU(2)_\ell$ An inclusion of fd $*$ algebras $A \subset B$ is encoded in a Bratteli diagram.

VI

Each is a sum of matrix algs and the diagram gives restriction rules for irreps. Assume diagram is connected. The Perron-Frobenius vector of the graph defines a trace tr on B (and A). Write $L^2(B)$ for B with ip $(x,y) = tr(xy^*)$, a bimodule over A acting by left and right multiplication.

$$\mathrm{End}\, L^2(B)_A = \langle B, e_1 \rangle \text{ where } e_1 = \text{orth proj onto } A \subset L^2(B)$$

Jones basic construction

This can be iterated
$$A_0 = A$$
$$A_1 = B$$
$$A_2 = \langle B, e_1 \rangle$$
$$A_3 = \langle B, e_1, e_2 \rangle$$
and so on.

The Bratteli diagram for the inclusions $A_i \subset A_{i+1}$ is obtained by reflection:

 ....

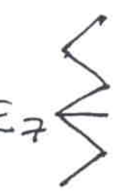
Get $e_i e_{i\pm1} e_i = \tau e_i$ where $\tau^{-1/2}$ is the PF norm of graph; in $SU(2)_\ell$ case $\tau = (q+q^{-1})^{-2}$, $q = e^{\pi i/\ell+2}$.

Set $M = \langle e_1, e_2, \dots \rangle''$ II_1 factor. Get bimodule $H_{1/2}$ by taking $L^2(M)$ with right action shifted by ρ, $\rho(e_i) = e_{i+1}$. Then find generated system of irreds, self conj, $H_0, H_{1/2}, \dots, H_{\ell/2}$ with

$$H_i \boxtimes H_j = \bigoplus_{\substack{|i-j| \le k \le i+j \\ i+j+k < \ell}} H_k \qquad SU(2)_\ell \text{ fusion rules}$$

Naturally a braided $\otimes$ category.

Building Quantum Subgroups

Take another D, E graph of norm < 2, eg E_6  or E_7  and repeat above construction. Let p be any minimal projection in A_0 (corr to one of 3 left vertices for E_6). Set

$$M = \langle e_1, e_2, \dots \rangle''.p \subseteq N = p(\cup A_n)''p.$$

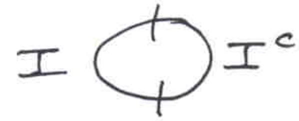
Then N is a quantum subgroup in the system of H_i's. For example for E_6, $N \cong H_0 \oplus H_3$ for approp. p. When take E_6, E_8, D_{even} get Abelian ergodic algebras in system. Then can define a (quantum) invariant theory subfactor. These correspond to the Jones subfactors of index < 4 and their invariant theory is described by the planar algebras of Jones.

(VII)

4. Boundary conditions in CFT (BCFT)

Look at system of bimodules associated to pe repns of loop groups $LG = C^\infty(S^1, G)$ or Diff S^1.

Take level ℓ p.e. reps of LG with G simple, s.c. Let $L_I G$ = loops trivial in I^c so conc. in I



For 2 pe reps X, Y set
$$\mathcal{X} = \mathrm{Hom}_{L_{I^c}G}(H_0, X)$$
$$\mathcal{Y} = \mathrm{Hom}_{L_I G}(H_0, Y)$$
with ip $\|x \otimes y\|^2 = (x^* x\, y^* y\, \Omega, \Omega)$. Fusion is computed by a <u>transport formula</u>. Any pe rep is unitarily equiv to H_0 on a local loop gp $L_I G$, so eg π_Y extends from $L_I G$ to $\pi_0(L_I G)''$. In partic $x^* x \in \pi_0(L_I G)''$, so $\pi_Y(x^* x)$ is defined. Taking x to be a smeared primary field (or more properly its phase), ~~to~~ the braiding relns for primary fields (= <u>monodromy props of KZ eqn</u>)
$$\Rightarrow \boxed{\pi_Y(x^* x) = \sum x_{kj}^* x_{kj}} \quad \text{TRANSPORT FORMULA}$$

where the sum is over primary fields $x_{kj} : Y = H_j \to H_k$ of charge $X = H_i$. This gives
$$H_i \boxtimes H_j \cong \oplus N_{ij}^k \otimes H_k$$
via $x \otimes y \mapsto \sum x_{kj}\, y\, \Omega$ where N_{ij}^k = dim of above space of p.f.s

There is extra structure here which won't describe – it makes the system into a braided $\otimes$ category. For $G = SU(2)$, get Jones category again, $H_0, H_{1/2}, \ldots, H_{\ell/2}$.

A conformal inclusion $LG \subset L\mathcal{G}$ is an inclusion $G \subset \mathcal{G}$ st every pe irrep of $L\mathcal{G}$ is finitely reducible on LG. This gives a finite depth subfactor $\pi_0(L_I G)'' \subset \pi_0(L_I \mathcal{G})''$ (AJW). More significantly K_0, vac rep of $L\mathcal{G}$, gives a <u>quantum subgp</u> in system of LG-modules.
$$m : K_0 \underset{LG}{\boxtimes} K_0 \longrightarrow K_0 \underset{L\mathcal{G}}{\boxtimes} K_0 = K_0$$
For $LSU(2)$, the conformal inclusions $SU(2)_{10} \subset SO(5)_1$, $SU(2)_{28} \subset (G_2)_1$ reproduce the q sgps E_6 and E_8. These examples were first studied by Feng Xu. For loop gps, all <u>AS</u>. q. sbgps arise this way.

The remaining non-Abelian q sgps (E_7 eg) come from (diagram) AMs of braided $\otimes$ categories (for E_7, this is D_{10}).

(VIII) Given an AM $\alpha \in \mathrm{Aut}(G)$, can change conjugation action of G on $C(G)$ to $g \cdot f(x) = f(g^{-1} x \alpha(g))$. As a G-module $C(G)_\alpha = \bigoplus V_i \otimes \alpha V_i^*$.

For an AM α of a braided $\otimes$ cat, can define a new algebra by $\bigoplus H_i \boxtimes \alpha H_i^*$

Decomposing this over classical H_0 - part, get ergodic non-comm algebras. One needs a generalisation of the Kazhdan-Wenzl reconstruction thm (for $SU(N)_\ell$) to prove that an AM of the fusion ring extends to an AM of the braided $\otimes$ cat.

(5) Existence for CTM Hamiltonian

Without entering into great detail we use the deformations of the data to define fusion for discrete series $c<1$ pe reps of the Virasoro algebra in the coset construction from $SU(2)_{\ell-1} \times SU(2)_1 \supset SU(2)_\ell$. Don't need the analogue of deformed Virasoro algebra only the primary fields which are directly constructed from those of the quantum affine algebra $U_q \widehat{sl}_2$ by coset construction. (Nevertheless Jimbo & Shiraishi have constructed q-Vir here despite no Segal-Sugawara)

Take primary fields $\varphi_{ki}(z): H_i \to H_k$ of charge $h_{2,1}$ of deformed Virasoro.

Then $\varphi_{ji}(1) \equiv \varphi_{ji}$ is bdd operator (Jimbo-Miwa)

For a path set $\varphi_I = \varphi_{i_N i_{N-1}} \cdots \varphi_{i_2 i_1}$

$i_1, i_2, i_3, \ldots, i_{N-1}, i_N$ (I)

Then $\varphi_J^* \varphi_I$ gives a repn of the Baxter algebra of observables $\bigcup_{n\geq 1} A_n$ in ABF model

The p.f.s satisfy $[L_0, \varphi(z)] = z\dfrac{d\varphi(z)}{dz} + \alpha\varphi(z)$

where L_0 = coset operator d. Also have braiding reln

$$\varphi_{ab}(z)\varphi_{bc}(w) = \sum_d \varphi_{ad}(w)\varphi_{dc}(z)\, W_{q,\frac{z}{w}}\begin{pmatrix} a & b \\ d & c\end{pmatrix}$$

from q KZ eqn of Frenkel-Reshetikhin. Taking partial derivs at $z=w=1$, find $\mathrm{ad}\, L_0 \equiv \mathrm{ad}\, H_{CTM}(q)$ on $\bigcup A_n$. So pe repns of $(\mathcal{A}, S)$ given by coset repns corr to $U_q \widehat{sl}_2$: Thus

$\boxed{\text{QUANTUM AFFINE ALG \& COSET CONSTR. UNIQUE SOLN TO ABF}}$