local CFTs (in two dimensions)

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Plan:

(I) The framework
(An approach to CFT using von Neum
algebras)

(II) Examples and results:
① Loop Groups;
② Orbifolds;
③ Cosets
(Kazama-Suzuki)

(I) Local CFT.

An incomplete history:

50-60's: Wightman axioms,

$\varphi(x)$. $x = (x^0, x^1, x^2, x^3)$

...

Given domain $D \subset \mathbb{R}^4$,

f, with $\operatorname{supp} f \subset D$,

$\varphi(f) := \int d^4x\, \varphi(x) f(x)$

generally unbounded ...

70's: Haag ...

Using "algebras of bounded operators"

So for $D \subset \mathbb{R}^4 \longrightarrow A(D)$

$A(D) \sim$ "von Neumann algebras generated by observables localized on D"

The relation between these two:

1. $\varphi(f)$, $\mathrm{supp} f \subset D$

   $\| \varphi(f) \|_s \leq c \| f \|_t$ ← Sobolev type

   $\varphi(f)$ has a closure.

   $\varphi(f)$ is "affiliated" with $\mathcal{A}(D)$;

2. $\mathcal{A}(D)$ is generated by "$\varphi(f)$'s, with $\mathrm{supp} f \subset D$"

More precisely:

$\varphi(f) = U \, |\varphi(f)|$ ← Polar decomposi[tion]

$|\varphi(f)|$ self-adjoint, U: bounded

$\{ \exp(it|\varphi(f)|), \ U, \ \mathrm{Supp} f \subset D \}''$

$= \mathcal{A}(D)$.

"$\varphi(x)$ is the "germ" of $\mathcal{A}(D)$'s".

CFT in two-dimension:

An adaption of Haag et. al's axioms to two dimension case. (Guido-Lon

A conformal precosheaf $\mathcal{A}$ of von Neuma algebras on the intervals of S' is a map: $S' \supset I \longrightarrow \mathcal{A}(I)$ on Hilbert space $\mathcal{H}$ that verifies:

A. Isotony. $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

B. Conformal Inv. There is a nontrival rep. U of $\widehat{PSL}(2,R)$ s.t.

$$U(g)\,\mathcal{A}(I)\,U(g)^* = \mathcal{A}(g.I), \quad \forall g;$$

C. locality: If $I_0 \cap I_1 = \phi$, then $[A(I_0), A(I_1)] = 0$;

D. Positivity of energy: the generator of the rotation subgroup $U(R)(\cdot)$ is ≥ 0

E. Existence of Vacuum there is a unit vector Ω which is $U(g)$ inv...

Remarks:

i) $\widehat{PSL(2,\mathbb{R})} \subset \widehat{\mathrm{Diff}}(S^1)$

(can add $\widehat{\mathrm{Diff}}(S^1)$ invariance.)

ii) F. <u>Irreducibility</u>

Ω is <u>unique</u> <u>up to scalar</u> multiple and $\bigvee_I \mathcal{A}(I)\,\Omega = \mathcal{H}$.

$F \Rightarrow \overline{\bigvee_I \mathcal{A}(I)} = B(\mathcal{H})$.

($\{I \mapsto \mathcal{A}(I)\}$ "bounded operator" version of VOA.)

A <u>representation</u> of

$\mathcal{A}$ is a family of representations

π_I of $\mathcal{A}(I)$ on $\mathcal{H}_\pi$ and a

unitary rep. U_π of $\widehat{PSL(2,\mathbb{R})}$

with positive energy such that

$$I_1 \supset I_0 \Rightarrow \pi_{I_1}\big|_{\mathcal{A}(I_0)} = \pi_{I_0} ;$$

$$U_\pi(g)\, \pi_I(x)\, U_\pi(g^*) = \pi_{g.I}(\, U(g)\, x\, U(g)^*).$$

$\forall x \in \mathcal{A}(I)$.

"equivalence classes of reps" $\leftarrow$ "Superselection sectors"

Fundamental Questions.

Study all (irr) rep. of $\mathcal{A}$.

<u>One advantage of $\mathcal{A}$</u>:

1) . Haag duality:

$$\mathcal{A}(I') = \mathcal{A}(I)'$$;

. All $\mathcal{A}(I)$ are type III_1 factor;

Let π be a rep. of $\mathcal{A}$.

$$\pi_I(A(I)) \cong A(I).$$

choose an identification.

so $\pi_I(x) = x, \ \forall x \in A(I)$.

Then $\left[\pi_{I'}(A(I')), \ \pi_I(A(I))\right] = 0$

$$\Rightarrow \pi_{I'}(A(I')) \subset A(I)' = A(I')$$

so: $\pi_{I'} \in \mathrm{End}(A(I'))$.

$\pi_{I'}(A(I')) \subset A(I')$: (subfactor with extremely rigid structures ... V. Jones ...)

<u>Fusion</u> of two reps π_1, π_2.

Fix I,

$\pi_{1,I'}$, $\pi_{2,I'} \in \mathrm{End}(A(I'))$

$\Rightarrow \pi_{1,I'} \cdot \pi_{2,I'} \in \mathrm{End}(A(I'))$

(In fact, comes from a rep $\pi_1 \boxtimes \pi_2$ of A.)

" Composition of charges "

or "Conne's Fusion"

special case of " Correspondence "

The Fusion is "manifestly" associative:

$$\pi_1 \cdot (\pi_2 \cdot \pi_3) = (\pi_1 \cdot \pi_2) \cdot \pi_3.$$

$\pi_1 \oplus \pi_2$: the usual direct sum of reps.

With $\boxtimes$, $\circledast$, Reps of $\mathcal{A}$ generate a ring

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Examples:

VOA $LSU(N)_k$, positive energy representations.

$\Rightarrow \mathcal{A}_{SU(N)_k}$, π_i ;

A. Wassermann (Inv. 98)

$\Rightarrow \{\pi_i\}$ generates Verlinde ring under fusion.

Kawahigashi - Longo - Müeger

and Xu

$\Rightarrow \{\pi_i\}$ are all the irrep of $\mathcal{A}_{SU(N)_k}$.

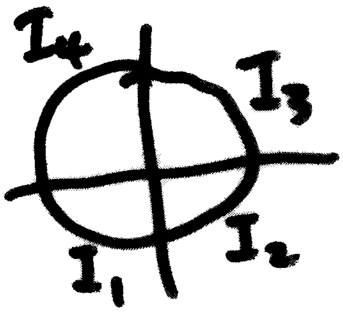
Def (KLM)

$\mathcal{A}(\cdot)$ is completely rational

if: (0) Splitting

(1) $\mathcal{A}(I)$ is strongly additive;

(2) $[\mathcal{A}((I_1 \vee I_3)')', \mathcal{A}(I_2 \vee I_4)] < \infty$



Theorem (KLM)

$\mathcal{A}(\cdot)$ is completely rational
$\Rightarrow$ $\mathcal{A}$ has only finitely many irrep.

Xu: $\mathcal{A}_{SU(N)_k}$ is c.r.

Orbifolds, Cosets

Why?

All known examples of rational CFT come from Lattices, Affine KM, and their orbifolds, cosets.

e.g. $V^\#$,

Vir (unitary) :

$SU(2)_{1+k} \subset SU(2)_k \times SU(2)$

(GKO) ...

Orbifolds

$\mathcal{A}$: conformal precosheaf on $\mathcal{H}$.

G: a finite group. $G \xrightarrow{\alpha} U(\mathcal{H})$ action

G acts **properly** on $\mathcal{A}$ if:

(1) $U(g)\,\mathcal{A}(I)\,U(g)^* = \mathcal{A}(I)$;

(2) $U(g)\Omega = \Omega$.

((1), (2) $\Rightarrow$ $[U(g), \widehat{PSL(2,\mathbb{R})}] = 0$)

$$\mathcal{A}^G(I) := \{a \in \mathcal{A}(I),\ U(g)aU(g)^* = a$$

$\mathcal{A}^G(I)$: orbifold of $\mathcal{A}$ w.r.t. G.

Theorem (2000)

If $\mathcal{A}$ is completely rational, then $\mathcal{A}^G$ is also c.r.
In fact $\mathcal{A}^G$ has only finitely many irreps and they generate a unitary Modular Tensor category c.f. Turaev's book

e.g. $U(1)^{\mathbb{Z}_2}$: $\mathbb{Z}_2 : \alpha \to -\alpha$

VOA reps of $U(1)^{\mathbb{Z}_2}$ Dong, Nagata

$\Rightarrow$ reps of $\mathcal{A}_{U(1)^{\mathbb{Z}_2}}$ .. Kac & Todo

Cor: ① If $G \neq \{1\}$
$\mathcal{A}^G$ must have irreps which don't come from restrictions of $\mathcal{A}$. (twisted reps)

② Let Untwist := the set of reps of $\mathcal{A}^G$ coming from the restrictions of $\mathcal{A}$. Then Untwist is closed under fusion.

Cosets.

$$H \subset G$$

$$LH \subset LG.$$

$$H^i = \bigoplus_\alpha H_{(i,\alpha)} \otimes H_\alpha$$

Define

$$\mathcal{A}_{G/H}(I) := \pi(L_I H)' \cap \pi(L_I G)$$

$H_{(i,\alpha)}$ naturally gives rise to reps of $\mathcal{A}_{G/H}$.

Question: Fusion of $\pi_{(i,\alpha)}$?

Unitary MTC?

Theorem (98)

Let $H = SU(N)_{k+l} \subset G = SU(N)_k \times SU(N)_l$

then $\mathcal{A}_{G/H}$ has only finitely many irrep, and they generate a unitary MTC. (There is also a formula for the corresponding closed 3-mfld invariants)

rk: (1) $N=2$, $l=1$, Virasoro series ...

(2) $N \geq 3$, $l=1$, W algebras...

(3) "Fixed-point resolutions"
$\Leftrightarrow$ rules for decomposing
$\pi_{(i,\alpha)}$ into irreducible
pieces.

<u>Cor</u>: Let $H = SU(N)_{k+1} \subset$
$G = SU(N)_k \times SU(N)_1$,
and $VOA(G/H)$ the coset
VOA. Then all $\pi_{(i,\alpha)}$
are irreducible.

No VOA proof so far for $N \geq 3$.

"Proof" of Cor.

Powerful machinery of subfactors (Ind / Res)

$\Rightarrow \operatorname{End}_{\mathcal{A}_{G/H}}(\pi(i,\alpha)) = \mathbb{C}\cdot 1\mathrm{d}$

$\Rightarrow H(i,\alpha)$ as rep of $\mathcal{A}_{G/H}$ is irreducible

$\Leftrightarrow H(i,\alpha)$ as rep of $VOA(G/H)$ is also irrep since $VOA(G/H)$ is the "germ" of $\mathcal{A}_{G/H}$.

<u>Technical definition</u>:

$LH \subset LG$ is said to be

<u>cofinite</u> if

$$\pi(L_I H)'' \vee (\pi(L_I H)' \cap \pi(L_I G)) \subset \pi(L_I G)''$$

has finite index.

<u>Rk</u>: 1) Independent of π, I;

2) Conj. to be true for all $LH \subset LG$.

Theorem (99): $\mathcal{L}H \lhd \mathcal{L}G$.

Assume $\mathcal{A}_G$, $\mathcal{A}_H$ are completely rational.

If $\mathcal{L}H \subset \mathcal{L}G$ is cofinite, then $\mathcal{A}_{G/H}$ has only finitely many irrep and they generate unitary MTC.

(In fact, every irrep of $\mathcal{A}_{G/H}$ appears in some $\pi_{(i,\alpha)}$)

Cor: Under the assumption of the above theorme, the Kac-Wakimoto Conjecture (Adv. Math. 1988) is true.

$$\left(\text{i.e. } \operatorname{Tr}_{H(i,\alpha)} e^{2\pi i \tau L_{G/H}(0)} \underset{\tau \to 0}{\sim} b_{(i,\alpha)}\, e^{-\cdots} \quad b_{(i,\alpha)} \neq 0 \right)$$

$\Rightarrow$ $\underline{\infty}$ series with no rep. theory proof.

"Proof" of the Cor.

Denote by $d(i,\alpha)$ the "statistical dimension" of $\pi_{(i,\alpha)}$.

From Thm.

$$d(i,\alpha) = \frac{b(i,\alpha)}{b(1,1) > 0} \geq 1$$

$$\Rightarrow \quad b(i,\alpha) \geq b(1,1) > 0$$

$\Rightarrow$ Kac-Wakimoto.

Kazama-Suzuki

Cosets bases on

$$Gr(n.m) = \frac{SU(n+m)}{SU(n)\times SU(m)\times U(1)}.$$

<u>Coset</u>:

$$\frac{G = SU(n+m)_k \times Spin(2nm)_1}{H = SU(n)_{m+k} \times SU(m)_{n+k} \times U(1)}.$$

$mn(m+n)\times(m+n+k)$

<u>e.g.</u> $n=m=1$, $\frac{SU(2)_k \times U(1)}{U(1)}$..

$m \to k$

$$\frac{SU(k+1)_1 \times Spin(2k)_1}{SU(k)_2 \times \cdots} \Rightarrow$$

$N=2$ Minimal model (Gepner ...)

$$c_{G/H} = \frac{3nmk}{n+m+k} > 1$$

<u>Symmetry</u> <u>(duality)</u> $n \leftrightarrow m \leftrightarrow k$

special

<u>S. Naulich, H. Schnitzer</u>:

$$\mathcal{A}_{G_1/H_1}, \quad \text{MTC } R \to \mathcal{A}_{G_2/H_2}$$

<u>Theorem</u> (math.QA/0108085)

(1) $\mathcal{A}_{G_1/H_1}$ is completely rational.

has only finitely many irrep. which generate a unitary MTC.

(2) $\mathcal{A}_{G_1/H_1} \cong \mathcal{A}_{G_2/H_2}$

Rk.

(1). In fact all irrep are determined. (i.e. $\pi_{(i,\alpha)}$ are decomposed into irreps in certain ways..)

(2). (2) of Thm implies all chiral quantities of A_{G_1/H_1} (like S, T-matrix, B, F data) are invariant under $m \leftrightarrow k$.

(3) ?? $VOA(G_1/H_1) \cong VOA(G_2/H_2)$