# On the Possible Group Theoretic and Low-dimensional Origins of Spacetime Supersymmetry

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There are some mathematical problems that lie at the foundation of the concept of spacetime supersymmetry that have remained unsolved for over thirty years. This simple fact seems almost a “secret” that goes largely unnoticed as the topic of supersymmetry has become more broadly investigated by communities of both mathematicians and physicists. As one who became interested in the topic soon after its introduction by Wess and Zumino in the western physics literature (the concept of supersymmetry appeared in the russian literature prior to this), I remain acutely aware of this situtaion.

Can it be that theoretical physics is capable of generating a problem of such difficulty so as to remain intractable for such a length of time? I suspect that it is more the fact that this problem remains one little investigated that explains this situation. One reason I was happy to receive this invitation to speak at the Mathematical Sciences Research Institute Workshop “Conformal Field Theory and Supersymmetry” was to have the opportunity to present this unsolved problem before an audience dominated by mathematicians. I am hopeful that the distinct perspective of this community with bring a fresh viewpoint to this problem that we physicists have not solved to this point.

In my efforts spent thinking about this problem, I have found what I believe are some tantalizing hints that may point toward some progress. Today in addition to the statement of this unsolved problem, I will also present two distinct observations that may lead to a set of tools that will become analogous to the root and weight spaces of Lie algebra. My observations come from two sources. One of these is the discovery of a link between a certain class of representations of real Clifford algebras and the realization of 1D supersymmetric models. Since it is real Clifford algebras which have been found to play this role, I have long suspected that this may lead to the additional importance of KO-theory making its appearance in such theories. The second obervation comes from the study of model-independent realizations of super Virasoro algebras.

Finally, as a physicist I am also acutely aware of the difficulty that comes about because of the differences in the two different languages used by the two distinct communities. I am not a native of “math-speak” and so accordingly I ask my audience’s indulgence as I try my best to relay via my natural language of “physics-speak” concepts that will be understood by all and especially to state the problem in a form that is understandable by members in both communities.

## (I.) A Statment of the Problem

Perhaps the most universal starting point to understand what is space time supersymmetry is to begin from the space of fields.

### Fields

$$\begin{array}{rcl} & & \quad \text{scalar} \quad \text{vector} \quad \text{graviton} \qquad\qquad \text{spinor} \quad \text{gravitino} \\ \mathcal{F} &=& \{ \ \phi(x)\,,\ A_{\underline{a}}(x)\,,\ h_{\underline{a}\,\underline{b}}(x), \dots \ \} \ \oplus \ \{ \ \ \lambda^{\alpha}(x)\,,\ \psi_{\underline{a}}{}^{\beta}(x), \dots \ \} \\ & & \text{spin}-0,\ \text{spin}-1,\ \text{spin}-2 \qquad\quad \text{spin}-1/2,\ \text{spin}-3/2 \\ \mathcal{F} &=& \{\mathcal{F}\}_b \ \oplus \ \{\mathcal{F}\}_f \end{array} \tag{1}$$

The space of fields $\mathcal{F}$ has a natural $\mathbf{Z}_2$ grading according to the representations into which the various fields fall.

Bosons

So for example, the "scalar" field $\phi(x)$ may be regarded as a map from some d-dimensional manifold with coordinates $x$ into the reals numbers. The "vector" field $A_{\underline{a}}(x)$ may be regarded as a map from some d-dimensional manifold with coordinates $x$ to the tangent vector space of the manifold. The graviton field $h_{\underline{a}\,\underline{b}}(x)$ may be regarded as a map from some d-dimensional manifold with coordinates $x$ to the symmetrical Cartesian product of the tangent vector space with itself.

Fermions

The "spinor" field $\lambda^{\alpha}(x)$ may be regarded as a map from some d-dimensional manifold with coordinates $x$ into the double cover of the tangent space to the manifold. The "gravitino" field $\psi_{\underline{a}}{}^{\beta}(x)$ may be regarded as a map from some d-dimensional manifold with coordinates $x$ into the Cartesian product of the double cover of the tangent space with the tangent space.

The supersymmetry transformation is a map that acts between the spaces $\{\mathcal{F}\}_b$ and $\{\mathcal{F}\}_f$. Physicists have long assumed that this map is homotopic to the identity map and thus assume the existence of an infinitesimal operator $\delta_Q$ which depends on a parameter $\epsilon^{\alpha}$ (also valued in the double cover) with the property

$$\delta_Q(\epsilon^{\alpha})\,\mathcal{F} \ = \ \{\widetilde{\mathcal{F}}\}_f \ \oplus \ \{\widetilde{\mathcal{F}}\}_b \tag{2}$$

The elements of $\{\widetilde{\mathcal{F}}\}_f$ are linear in $\epsilon^\alpha$ and linear in the elements of $\{\mathcal{F}\}_f$ and may involve tensors that are invariant under the action of isometries of the metric of the d-dimensional manifold and can involve first derivatives. The elements of $\{\widetilde{\mathcal{F}}\}_b$ are linear in $\epsilon^\alpha$ and linear in the elements of $\{\mathcal{F}\}_b$ and may involve tensors that are invariant under the action of isometries of the metric of the d-dimensional manifold and can involve first derivatives.

There are another set of infinitesimal variations (called "translations" by physicists) that can be defined on the space of fields. These are denoted by $\delta_P$ and depend on parameters $\xi^{\underline{a}}$ where these parameters are valued in the tangent space to the manifold.

$$\delta_P(\xi^{\underline{a}})\,\mathcal{F} \;=\; (\xi^{\underline{a}}\partial_{\underline{a}}\{\mathcal{F}\}_b) \;\oplus\; (\xi^{\underline{a}}\partial_{\underline{a}}\{\mathcal{F}\}_f) \tag{3}$$

Physicists say that a system which consists of a subset of all the fields is supersymmetric if the system admits the following equation.

$$\delta_Q(\epsilon_1^\alpha)\,\delta_Q(\epsilon_2^\beta) \;-\; \delta_Q(\epsilon_2^\alpha)\,\delta_Q(\epsilon_1^\beta) \;=\; \delta_P(\xi^{\underline{a}}) \quad , \tag{4}$$

where $\xi^{\underline{a}} \;=\; i2 < \epsilon_1^\alpha \gamma^{\underline{a}}{}_{\alpha\beta} \epsilon_2^\beta >$. Systems satisfying this condition are said to be "off-shell supersymmetric" or to possess "off-shell spacetime supersymmetry." The remarkable fact is that even now thirty years after its first statement, the general solution to this problem is still *not* known.

What the physics community has quite effectively used is the fact that there is a related set of equations on the space of fields that is simpler to solve.

A hypersurface in field space may be defined by imposing some differential equations on the fields. For example, the scalar field might be harmonic, satisfying the condition that its d'Alembertian vanishes. In physics we call such a condition "an equation of motion" if it is derivable by the extremization of some function, typically denoted by $S$, that we call the action. Let us denote such equations of motion generically by the symbol $\partial S$. Most of the discussions in the physics literature involve representations such that

$$\delta_Q(\epsilon_1^\alpha)\,\delta_Q(\epsilon_2^\beta) \;-\; \delta_Q(\epsilon_2^\alpha)\,\delta_Q(\epsilon_1^\beta) \;=\; \delta_P(\xi^{\underline{a}}) \;+\; \partial S \tag{5}$$

So why should finding off-shell representations be so difficult? In truth no one knows. But it does seem to be a problem worthy of study.

This section will be closed with a discussion of a very well known example of the points raised above. There is one such representation known as "4D, $N = 8$ supergravity." This representation uses all of the fields explicitly noted in our introductory remarks above where the "spins" of the fields satisfy $0 \le s \le 2$. However, in this representation, there is not a single supersymmetry parameter, but instead eight such parameters occur, (i. e. $\epsilon^{\alpha} \to \epsilon^{\alpha \rm I}$ with $\rm I = 1 , \dots 8$).

The fields of this representation are given by

**Table 1: Fields of 4D, $N = 8$ Supergravity**

| Field | Spin | Multiplicity |
|---|---|---|
| $h_{\underline{a}\,\underline{b}}$ | 2 | 1 |
| $\psi_{\underline{a}}{}^{\beta \, \rm I}$ | 3/2 | 8 |
| $A_{\underline{a}}{}^{\rm I\,J}$ | 1 | 28 |
| $\lambda_{\alpha}{}^{\rm I\,J\,K}$ | 1/2 | 56 |
| $\phi^{\rm I\,J\,K\,L}$ | 0 | 70 |

The multiplities for the fields can be seen to follow from the realization that the indices which count the multiplicities are similar to the indices on forms, i.e. possess a skew-symmetric property. The numbers 8, 28, 56 thus immediately follow as consequences. The number 70 follows from the fact that although $\phi^{\rm I\,J\,K\,L}$ is complex, in an eight dimensional space there exist a Levi-Civita tensor, which may be used to make the 4-form $\phi^{\rm I\,J\,K\,L}$ either self-dual or anti-self-dual. The explicit forms of the on-shell supersymmetry variations as well as the action $S$ for 4D, $N = 8$ supergravity can be found in the physics literature.

## (II.) Two Proposals for the Fundamental Study of Problem

I certainly am in no position to give the solution to this problem today. I would like to suggest however, that progress toward this solution may be emerging from two quite different sources,

(a.) Clifford Algebra Induced 1D Supersymmetry Representations

(b.) Model-independent Representations of Super Virasoro algebras.

In the remainder of my presentation, I wish to describe why I believe that the general problem discribed previously may have a resolution involving these ideas.

Since both of these suggestions involve systems with either one or two bosonic coordinates, one obvious objection that can be raised is that the problem of off-shell supersymmetric representations in higher d-dimensional manifolds cannot be related to such simple one and two dimensional systems.

The resolution to this, I believe lies in the fact that we physicists are quite used to relating higher dimensional field theories to lower dimensional ones via a technique we call reduction on a cylinder. In particular the fields of a high dimensional theory, may be studied in a projection where all space-like coordinates are regarded as the coordinates of some fiber and then we retain only the dependence of all fields on their temporal coordinates.

I suggest this observation allows the belief that there may exist ways to encode all the structures of the higher dimensional theories in terms of 1D and 2D systems. This may be akin to the fact that root and weight spaces are much simpler than the manifold of group coordinates and yet the former are completely capable of encoding all essential information about the local structure of Lie algebras.

## (III.) Clifford Algebra Induced 1D SUSY Reps

The simplest venue in which to study the phenomenon above is in the context wherein the fields are, in fact, simply functions of a single real parameter denoted by $\tau$ which we may assume takes on values between zero and one.

There is a special class of Clifford algebras that are closely related to 1D supersymmetric representations. The special class is defined as follows.

Introduce two real d-dimensional vector spaces denoted by $\mathcal{V}_L$ and $\mathcal{V}_R$.

Define spaces of all linear maps such that

$$\{\mathcal{M}\}_L : \mathcal{V}_L \to \mathcal{V}_R \quad , \quad \{\mathcal{M}\}_R : \mathcal{V}_R \to \mathcal{V}_L \quad ,$$
$$\{\mathcal{U}\}_L : \mathcal{V}_L \to \mathcal{V}_L \quad , \quad \{\mathcal{U}\}_R : \mathcal{V}_R \to \mathcal{V}_R \quad .$$

so that a composition of the maps satisfies,

$$\{\mathcal{M}\}_R \times \{\mathcal{M}\}_L : \mathcal{V}_L \to \mathcal{V}_L \quad ,$$
$$\{\mathcal{M}\}_L \times \{\mathcal{M}\}_R : \mathcal{V}_R \to \mathcal{V}_R \quad .$$

$\mathcal{GR}(\mathrm{d}, N)$ Algebras: A Special Class of Clifford Algebras

Definition:
Choose $N$ elements from $\{\mathcal{M}\}_L$ and $N$ elements from $\{\mathcal{M}\}_R$. If $\{\mathcal{M}(\mathrm{L}_{\mathrm{I}})\}_L$ (for a fixed number I) denotes the I-th element of $\{\mathcal{M}\}_L$ and $\{\mathcal{M}(\mathrm{R}_{\mathrm{K}})\}_R$ (for a fixed number K) denotes the K-th element of $\{\mathcal{M}\}_R$ then we require

$$
\begin{aligned}
[\,\{\mathcal{M}(\mathrm{R}_{\mathrm{I}})\}_R \times \{\mathcal{M}(\mathrm{L}_{\mathrm{K}})\}_L \ + \ \{\mathcal{M}(\mathrm{R}_{\mathrm{K}})\}_R \times \{\mathcal{M}(\mathrm{L}_{\mathrm{I}})\}_L\,] : \mathcal{V}_L \ &= \ -2\,\delta_{\mathrm{IK}}\mathcal{V}_L \quad , \\
[\,\{\mathcal{M}(\mathrm{L}_{\mathrm{I}})\}_L \times \{\mathcal{M}(\mathrm{R}_{\mathrm{K}})\}_R \ + \ \{\mathcal{M}(\mathrm{L}_{\mathrm{K}})\}_L \times \{\mathcal{M}(\mathrm{R}_{\mathrm{I}})\}_R\,] : \mathcal{V}_R \ &= \ -2\,\delta_{\mathrm{IK}}\mathcal{V}_R \quad ,
\end{aligned}
$$

for all possible choices $1 \le \mathrm{I}, \mathrm{K} \le N$. The subsets of $\{\mathcal{M}\}_L$ and of $\{\mathcal{M}\}_R$ that satisfy these condition, may be called "$\mathcal{GR}(\mathrm{d}, N)$ algebras."

For a fixed value of $N$, faithful matrix reps occur when $d \ge d_M$

$$
d_M \ = \ 2^{4m+1}\, F_{\mathcal{RH}}(r) \quad ,
$$

where $F_{\mathcal{RH}}(N)$ is the Radon-Hurwitz function, $N = 8m + r$ and $m \in Z$ (where if $N = 8$, then $m \equiv 0$).

Embedding $\mathcal{GR}(\mathrm{d}, N)$ Algebras in Standard Clifford Algebras

Let $\gamma^{\mathrm{I}}$ denote a standard rep of an element in a real Clifford algebra, i.e. under matrix maultiplication

$$
\gamma^{\mathrm{I}}\gamma^{\mathrm{J}} \ + \ \gamma^{\mathrm{J}}\gamma^{\mathrm{I}} \ = \ -\ 2\delta^{\mathrm{IJ}}\,\mathbf{I} \quad .
$$

pick only those Clifford algebras that admit an object denoted by $Q$ such that

$$
Q^2 \ = \ 1 \quad , \quad \gamma^{\mathrm{I}}\,Q \ + \ Q\,\gamma^{\mathrm{I}} \ = \ 0 \quad .
$$

Define "projectors" $P_{\pm} \equiv \frac{1}{2}[\mathbf{I} \pm Q]$ implying $P_+^2 = P_+$, $P_-^2 = P_-$, $P_{\pm}P_{\mp} = P_{\mp}P_{\pm} = 0$ and $P_+ + P_- = \mathbf{I}$,

Define $\{\Gamma\}$ via

$$
\{\Gamma\} \ \equiv \ (\,\mathbf{I},\, \gamma^{\mathrm{I}}_1,\, \gamma^{\mathrm{I}}_1 \wedge \gamma^{\mathrm{I}_2},\, \gamma^{\mathrm{I}}_1 \wedge \gamma^{\mathrm{I}_2} \wedge \gamma^{\mathrm{I}_3},\, \dots\, , \gamma^{\mathrm{I}_1} \wedge \cdots \wedge \gamma^{\mathrm{I}_N}\,)
$$

then the representation of the elements of $\{\mathcal{M}\}_L$, $\{\mathcal{M}\}_R$, $\{\mathcal{U}\}_L$ and $\{\mathcal{U}\}_R$ can be put into a one-to-correspondence such that

$$
\begin{aligned}
\{\mathcal{M}\}_L = P_+\{\Gamma\}P_- \quad , \quad \{\mathcal{M}\}_R = P_-\{\Gamma\}P_+ \quad , \\
\{\mathcal{U}\}_L = P_+\{\Gamma\}P_+ \quad , \quad \{\mathcal{U}\}_R = P_-\{\Gamma\}P_- \quad .
\end{aligned}
$$

Let $\phi_a$ denote the component of an element $\phi \in \mathcal{V}_L$ and $\psi_{\hat{a}}$ denote the component of an element $\psi \in \mathcal{V}_R$ be functions of a single parameter real $\tau$.

Define infinitesimal variations of these quantities by

$$\delta_Q(\epsilon^{\rm I})\phi_a \ = \ i\epsilon^{\rm I}\,({\rm L}^{\rm I})_a{}^{\hat{b}}\psi_{\hat{b}} \quad , \quad \delta_Q(\epsilon^{\rm I})\psi_{\hat{b}} \ = \ \epsilon^{\rm I}\,({\rm R}^{\rm I})_{\hat{b}}{}^a\partial_\tau\phi_a$$

where $({\rm L}^{\rm I})_a{}^{\hat{b}}$ and $({\rm R}^{\rm I})_{\hat{b}}{}^a$ respectively denote the matrix elements of the representations of ${\rm L}^{\rm I}$ and ${\rm R}^{\rm I}$. Then on all elements in $\mathcal{V}_L$ and $\mathcal{V}_R$ the following condition is satisfied

$$[\, \delta_Q(\epsilon_1^{\rm I}) \ , \ \delta_Q(\epsilon_2^{\rm I})\,] \ = \ i2\,\epsilon_1^{\rm I}\,\epsilon_2^{\rm I}\,\partial_\tau$$

This is a one dimensional off-shell representation of supersymmetry.

## Two Conjectures

Note that in the case where $N = 8$, the physical states of 4D, $N = 8$ supergravity are in one-to-one correspondence with the elements of $\{\mathcal{M}\}_L$, $\{\mathcal{M}\}_R$, $\{\mathcal{U}\}_L$ and $\{\mathcal{U}\}_R$.

Conjecture: All states in all spacetime supersymmetrical theories provide reps of the forms of the $\mathcal{GR}({\rm d}, N)$ enveloping algebras.

This simple conjecture does not take into account the spacetime spin of the states. So we believe there is more to the origin of spacetime supersymmetry that is related to 1D representations of super Virasoro algebras. It particular if we look back at the 4D, $N = 8$ representation, we simply observe that there is a relation between the spin $S$ of the field and the degree of its $\mathcal{GR}({\rm d}, N)$ $p$-form index.

$$S \ = \ \tfrac{1}{2}\,(\ 4 - p\ )$$

This leads to our second conjecture of the origin of spacetime supersymmmetry.

Conjecture: The spin of all states in all spacetime supersymmetrical theories are associated with their being realization of super-Virasoro algebras.

We will spend the rest of presentation in an attempt to present evidence for this.

## (IV.) Relations to 1D Infinite Dimensional Lie Algebras

The $N$-extend Real SuperLine

Define a real coordinate $\tau$ taking its values in the range (0,1).

Introduce real Grassmann numbers $\zeta^{\rm I}$, with ${\rm I} = 1, \dots, N$.

$$\zeta^{\rm I}\,\zeta^{\rm J} \;+\; \zeta^{\rm I}\,\zeta^{\rm J} \;=\; 0$$

Let $(\tau,\, \zeta^{\rm I})$ defines a "superpoint."

Since differentiation with regard to superfunctions can be defined, supervector fields can also be defined

$$\mathcal{W}(1|N) \;\equiv\; f(\tau,\,\zeta)\,\partial_\tau \;+\; g^{\rm I}(\tau,\,\zeta)\,\partial_{\rm I}$$

where

$$\partial_\tau \;\equiv\; \frac{\partial}{\partial\tau} \quad , \quad \partial_{\rm I} \;\equiv\; \frac{\partial}{\partial\zeta^{\rm I}}$$

Under graded commutation, the set of all such objects with suitable differentiation restriction on the coefficients form a closed set.

### A SuperVector Field Basis

$$\begin{aligned}
G_{\mathcal{A}}{}^{\rm I} &\equiv\; i\,\tau^{\mathcal{A}+\frac{1}{2}}\left[\,\partial^{\rm I} \;-\; i\,2\,\zeta^{\rm I}\,\partial_\tau\,\right] \;+\; 2(\,\mathcal{A} \,+\, \tfrac{1}{2}\,)\tau^{\mathcal{A}-\frac{1}{2}}\zeta^{\rm I}\zeta^{\rm K}\,\partial_{\rm K} \quad , \\
L_{\mathcal{A}} &\equiv\; -\left[\tau^{\mathcal{A}+1}\partial_\tau \;+\; \tfrac{1}{2}(\mathcal{A} \,+\, 1)\,\tau^{\mathcal{A}}\zeta^{\rm I}\,\partial_{\rm I}\,\right] \quad , \\
T_{\mathcal{A}}^{\rm I\,J} &\equiv\; \tau^{\mathcal{A}}\left[\,\zeta^{\rm I}\,\partial^{\rm J} \;-\; \zeta^{\rm J}\,\partial^{\rm I}\,\right] \quad , \\
U_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_q} &\equiv\; i\,(i)^{[\frac{q}{2}]}\,\tau^{(\mathcal{A}-\frac{(q-2)}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_{q-1}}\,\partial^{{\rm I}_q} \quad , \quad q \;=\; 3,\,\dots\,,\,N \,+\, 1 \quad , \\
R_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_p} &\equiv\; (i)^{[\frac{p}{2}]}\,\tau^{(\mathcal{A}-\frac{(p-2)}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_p}\,\partial_\tau \quad , \quad p \;=\; 2,\,\dots\,,\,N \quad ,
\end{aligned}$$

**Algebra of in this Basis**

$$
\begin{aligned}
[\, L_{\mathcal{A}} \,,\, L_{\mathcal{B}} \,\} &= \; (\, \mathcal{A} \,-\, \mathcal{B} \,)\, L_{\mathcal{A}+\mathcal{B}} + \tfrac{1}{8}\, c\, (\mathcal{A}^3 - \mathcal{A})\, \delta_{\mathcal{A}+\mathcal{B},0} \quad , \\
[\, L_{\mathcal{A}} \,,\, U_{\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \,\} &= \; -\, [\, \mathcal{B} \,+\, \tfrac{1}{2}\, (m-2)\, \mathcal{A} \,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \quad , \\
[\, G_{\mathcal{A}}{}^{\mathrm{I}} \,,\, G_{\mathcal{B}}{}^{\mathrm{J}} \,\} &= \; -\, i\, 4\, \delta^{\mathrm{I\,J}} L_{\mathcal{A}+\mathcal{B}} \;-\; i2(\mathcal{A}-\mathcal{B})\, [\, T_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}} \;+\; 2(\mathcal{A}+\mathcal{B})\, U_{\mathcal{A}+\mathcal{B}\mathrm{K}}^{\mathrm{I\,J\,K}} \,] \\
&\quad -\, ic(\mathcal{A}^2 - \tfrac{1}{4})\, \delta_{\mathcal{A}+\mathcal{B},0}\, \delta^{\mathrm{I\,J}} \quad , \\
[\, L_{\mathcal{A}} \,,\, G_{\mathcal{B}}{}^{\mathrm{I}} \,\} &= \; (\, \tfrac{1}{2}\mathcal{A} \,-\, \mathcal{B} \,)\, G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{I}} \quad , \\
[\, L_{\mathcal{A}} \,,\, R_{\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \,\} &= \; -\; [\, \mathcal{B} \,+\, \tfrac{1}{2}\, (m-2)\, \mathcal{A} \,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \\
&\quad -\; [\, \tfrac{1}{2}\, \mathcal{A}\, (\mathcal{A}+1) \,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m\, J}{}_{J} \quad , \\
[L_{\mathcal{A}} \,,\, T_{\mathcal{B}}^{\mathrm{I\,J}}\} &= \; -\, \mathcal{B}\, T_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}} \\
[\, R_{\mathcal{A}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \,,\, R_{\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_n} \,\} &= \; -\, (i)^{\sigma(mn)}\, [\, \mathcal{A} \,-\, \mathcal{B} \,-\, \tfrac{1}{2}(m \,-\, n) \,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m\, \mathrm{J}_1 n \cdots \mathrm{J}_n} \quad , \\
[\, T_{\mathcal{A}}^{\mathrm{I\,J}} \,,\, T_{\mathcal{B}}^{\mathrm{K\,L}} \,\} &= \; T_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,K}}\, \delta^{\mathrm{JL}} +\; T_{\mathcal{A}+\mathcal{B}}^{\mathrm{J\,L}}\, \delta^{\mathrm{IK}} -\; T_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,L}}\, \delta^{\mathrm{JK}} -\; T_{\mathcal{A}+\mathcal{B}}^{\mathrm{J\,K}}\, \delta^{\mathrm{IL}} \\
&\quad +\; \tilde{c}\, (\mathcal{A}-\mathcal{B})(\delta^{\mathrm{IK}}\delta^{\mathrm{JL}} \,-\, \delta^{\mathrm{IL}}\delta^{\mathrm{JK}})\delta_{\mathcal{A}+\mathcal{B},0} \quad , \\
[\, G_{\mathcal{A}}{}^{\mathrm{I}} \,,\, R_{\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_m} \,\} &= \; 2\, (i)^{\sigma(m)}\, [\, \mathcal{B} \,+\, (m-1)\, \mathcal{A} \,+\, \tfrac{1}{2} \,]\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}_1 \cdots \mathrm{J}_m} \\
&\quad -\; (i)^{\sigma(m)} \sum_{r=1}^{m} (-1)^{r-1}\, \delta^{I\,\mathrm{J}_r}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_{r-1}\, \mathrm{J}_{r+1} \cdots \mathrm{J}_m} \\
&\quad -\; (-i)^{\sigma(m)}\, [\, \mathcal{A} \,+\, \tfrac{1}{2} \,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_m\, \mathrm{I}} \\
&\quad +\; 2\, (i)^{\sigma(m)}\, [\, \mathcal{A}^2 \,-\, \tfrac{1}{4} \,]\, U_{\mathcal{A}+\mathcal{B}}^{I\, \mathrm{J}_1 \cdots \mathrm{J}_m\, K}{}_{K} \quad , \\
[\, G_{\mathcal{A}}{}^{\mathrm{I}} \,,\, U_{\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_m} \,\} &= \; 2\, (i)^{\sigma(m)}\, [\, \mathcal{B} \,+\, (m-2)\, \mathcal{A} \,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,J}_1 \cdots \mathrm{J}_m} \\
&\quad -\; 2\, (-i)^{\sigma(m)}\, [\, \mathcal{A} \,+\, \tfrac{1}{2} \,]\, \delta^{\mathrm{I\,J}_m}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_{m-1} K}{}_{K} \\
&\quad -\; (i)^{\sigma(m)} \sum_{r=1}^{m-1} (-1)^{r-1}\, \delta^{\mathrm{I\,J}_r}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_{r-1}\, \mathrm{J}_{r+1} \cdots \mathrm{J}_m} \\
&\quad +\; 2\, (-i)^{\sigma(m)}\, \delta^{\mathrm{I\,J}_m}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_{m-1}} \quad , \\
[\, R_{\mathcal{A}}^{\mathrm{I}_1 \cdots \mathrm{I}_m} \,,\, U_{\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_n} \,\} &= \; (-i)^{\sigma(mn)} \sum_{r=1}^{m} (-1)^{r-1}\, \delta^{\mathrm{I}_r\, \mathrm{J}_n}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1 \cdots \mathrm{J}_{n-1} \mathrm{I}_1 \cdots \mathrm{I}_{r-1}\, \mathrm{I}_{r+1} \cdots \mathrm{I}_m} \\
&\quad +\; i(i)^{\sigma(mn)}\, [\, \mathcal{B} \,-\, \tfrac{1}{2}(n-2) \,]\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1 \cdots \mathrm{I}_m\, \mathrm{J}_1 \cdots \mathrm{J}_n} \quad ,
\end{aligned}
$$

$$[\, U_{\mathcal{A}}^{\mathrm{I}_1\cdots\mathrm{I}_m}\,,\, U_{\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_n}\,\} = \; -\,(i)^{\sigma(mn)}\Big\{\sum_{r=1}^{m}(-1)^{r-1}\,\delta^{\mathrm{I}_m\mathrm{J}_r}\; U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_{m-1}\,\mathrm{J}_1\cdots\mathrm{J}_{r-1}\,\mathrm{J}_{r+1}\cdots\mathrm{J}_{n-1}\mathrm{J}_n}$$

$$-\,(-1)^{mn}\sum_{r=1}^{m}(-1)^{r-1}\,\delta^{\mathrm{I}_r\mathrm{J}_n}\; U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J}_1\cdots\mathrm{J}_{n-1}\,\mathrm{I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_{m-1}\mathrm{I}_m}\Big\} \quad ,$$

$$[\, T_{\mathcal{A}}^{\mathrm{I\,J}}\,,\, G_{\mathcal{B}}{}^{\mathrm{K}}\,\} \;=\; 2\,(\delta^{\mathrm{JK}}G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{I}} - \delta^{\mathrm{IK}}G_{\mathcal{A}+\mathcal{B}}{}^{\mathrm{J}})$$

$$+\;\; 2\mathcal{A}\,(\delta^{\mathrm{JK}}\; U_{\mathcal{A}+\mathcal{B}\mathrm{L}}^{\mathrm{I\,,L}} - \delta^{\mathrm{IK}}\; U_{\mathcal{A}+\mathcal{B}\mathrm{L}}^{\mathrm{J\,L}} + U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J\,K\,I}} - \; U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,K\,J}}) \quad ,$$

$$[\, T_{\mathcal{A}}^{\mathrm{I\,J}}\,,\, R_{\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_p}\,\} = \;\sum_{r=1}^{p}(-1)^{r+1}\,(\delta^{\mathrm{J\,I}_r}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_p} - \delta^{\mathrm{I\,I}_r}\, R_{\mathcal{A}+\mathcal{B}}^{\mathrm{J\,I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_p})$$

$$+\, i\,(-1)^{p}\mathcal{A}\,(U_{\mathcal{A}+\mathcal{B}-1}^{\mathrm{I\,I}_1\cdots\mathrm{I}_\mathrm{p}\,\mathrm{J}} - U_{\mathcal{A}+\mathcal{B}-1}^{\mathrm{J\,I}_1\cdots\mathrm{I}_\mathrm{p}\,\mathrm{I}}) \quad ,$$

$$[\, T_{\mathcal{A}}^{\mathrm{I\,J}}\,,\, U_{\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_p}\,\} = \;\sum_{r=1}^{p}(-1)^{r+1}\,(\delta^{\mathrm{J\,I}_r}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{I\,I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_p} - \delta^{\mathrm{I\,I}_r}\, U_{\mathcal{A}+\mathcal{B}}^{\mathrm{J\,I}_1\cdots\mathrm{I}_{r-1}\,\mathrm{I}_{r+1}\cdots\mathrm{I}_p})$$

$$+\;\,(\delta^{\mathrm{I}_\mathrm{p}\,\mathrm{I}}\, U_{\mathcal{A}+\mathcal{B}-1}^{\mathrm{I}_1\cdots\mathrm{I}_{\mathrm{p}-1}\,\mathrm{J}} - \;\delta^{\mathrm{I}_\mathrm{p}\,\mathrm{J}}\, U_{\mathcal{A}+\mathcal{B}-1}^{\mathrm{I}_1\cdots\mathrm{I}_{\mathrm{p}-1}\,\mathrm{I}}\,) \quad ,$$

where the function $\sigma(m) = 0$ if $m$ is even and $-1$ if $m$ is odd. Here the central extensions $c$ and $\tilde{c}$ are unrelated since we have only imposed the Jacobi identity.

## The Maximal Primary Basis

Definition:
If $\mathcal{F}_{\mathcal{A}}^{\mathrm{I}_1\cdots\mathrm{I}_\mathrm{p}}$ is a primary generator, then there must exist some particular mode-dependent coefficient $\lambda$ such that this generator satisfies

$$[L_{\mathcal{A}},\, \mathcal{F}_{\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_\mathrm{p}}\,] \;=\; -\lambda(\mathcal{A},\mathcal{B},\mathrm{p})\,\mathcal{F}_{\mathcal{A}+\mathcal{B}}^{\mathrm{I}_1\cdots\mathrm{I}_\mathrm{p}} \quad ,$$

for fixed number of indices p.

Example:
$L_{\mathcal{A}}$ is a primary operator.

$$[\; L_{\mathcal{A}}\,,\, L_{\mathcal{B}}\;\} \;=\; (\,\mathcal{A}\,-\,\mathcal{B}\,)\,L_{\mathcal{A}+\mathcal{B}}$$

$$
\begin{aligned}
G_{\mathcal{A}}{}^{\rm I} &\equiv\; i\,\tau^{\mathcal{A}+\frac{1}{2}}\Big[\,\partial^{\rm I}\;-\;i\,2\,\zeta^{\rm I}\,\partial_\tau\,\Big]\;+\;2(\,\mathcal{A}\,+\,\tfrac{1}{2}\,)\tau^{\mathcal{A}-\frac{1}{2}}\zeta^{\rm I}\zeta^{\rm K}\,\partial_{\rm K}\quad ,\\
L_{\mathcal{A}} &\equiv\;-\,\Big[\,\tau^{\mathcal{A}+1}\partial_\tau\;+\;\tfrac{1}{2}(\mathcal{A}\,+\,1)\,\tau^{\mathcal{A}}\zeta^{\rm I}\,\partial_{\rm I}\,\Big]\quad ,\\
T_{\mathcal{A}}^{\rm I\,J} &\equiv\;\tau^{\mathcal{A}}\Big[\,\zeta^{\rm I}\,\partial^{\rm J}\;-\;\zeta^{\rm J}\,\partial^{\rm I}\,\Big]\quad ,\\
U_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_q} &\equiv\; i\,(i)^{[\frac{q}{2}]}\,\tau^{(\mathcal{A}-\frac{(q-2)}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_{q-1}}\,\partial^{{\rm I}_q}\quad ,\quad q\;=\;3,\,\dots\,,\,N\,+\,1\quad ,\\
\mathcal{R}_{\mathcal{A}}^{{\rm I}_1\cdots{\rm I}_p} &\equiv\;(i)^{[\frac{p}{2}]}\,\tau^{(\mathcal{A}-\frac{p}{2})}\zeta^{{\rm I}_1}\cdots\,\zeta^{{\rm I}_p}\,[\,\tau\partial_\tau\;+\;(\frac{\mathcal{A}+1}{p-2})\zeta^{L}\partial_{L}\,]\quad ,\quad p\;=\;3,\,\dots\,,\,N\quad ,\\
R_{\mathcal{A}}^{\rm IJ} &\equiv\; i\,\tau^{\mathcal{A}}\zeta^{\rm I}\zeta^{\rm J}\,\partial_\tau\qquad ,
\end{aligned}
$$

possesses only one non-primary generator, namely $R_{\mathcal{A}}^{\rm IJ}$. We will refer to this basis as the "maximal primary basis" for the $\mathcal{GR}$ super-Virasoro algebra.

**Abstracting Fields and Transformations**

Since $\mathcal{L}$ is the generator of diffeomorphisms we can use its action on the other generators to determine the tensor properties of the fields. Fields in this contect are simply regarded as "vectors" in the infinite-dimensional Lie algebra Let

$$
\mathcal{L}' =\;\Big(\,L_\xi,\;G_{\chi^{\rm I}}{}^{\rm I},\;T_{t^{\rm J\,K}}^{\rm J\,K},\;\oplus_{\{\rm I_q\}}\,U_{\mu^{\{\rm I_q\}}}^{\{\rm I_q\}},\;\oplus_{\{\rm J_q\}}\,R_{r^{\{\rm J_q\}}}^{\{\rm J_q\}};\,\alpha\Big)\quad ,
$$

represent the generators with generic functions and $\oplus_{\{\rm I_q\}}$ represents the direct sum over all distinct generators. Then from the algebra we see that

$$
\begin{aligned}
[\;(L_\xi,\alpha)\,,\,(L_\zeta,\beta)\;\} &=\;(L_{\xi'\zeta-\xi\zeta'},\;\tfrac{c}{i2\pi}\int(\xi''\zeta'-\zeta''\xi')d\,x)\quad ,\\
[\;L_\xi\;,\;G_{\chi^{\rm I}}^{\rm I}\;\} &=\;G^{\rm I}_{(-\xi(\chi^{\rm I})'+\frac{1}{2}\xi'\chi^{\rm I})}\quad ,\\
[\;L_\xi\;,\;T_{t^{\rm RS}}^{\rm RS}\;\} &=\;T^{\rm RS}_{(-\xi\,(t^{\rm RS})')}\quad ,\\
[\;L_\xi\;,\;U_{w^{\{\rm V_r\}}}^{\{\rm V_r\}}\;\} &=\;U^{\{\rm V_r\}}_{(-\xi(w^{\{\rm V_r\}})'-\frac{1}{2}(r-2)\xi' w^{\{\rm V_r\}})}\quad ,\\
[\;L_\xi\;,\;R_{\rho^{\{\rm T_r\}}}^{\{\rm T_r\}}\;\} &=\;R^{\{\rm T_r\}}_{(-(\rho^{\{\rm T_r\}})'\xi-\frac{1}{2}(r-2)\xi'(\rho^{\{\rm T_r\}}))}-\tfrac{i}{2}\,U^{\{\rm T_r\}}_{(\xi''\rho^{\{\rm T_r\}})}\quad ,
\end{aligned}
$$

**All $*$ Commutators**:

$$L_\xi * (\bar{L}_D, \bar{\beta}) = \bar{L}_{\tilde{D}} \quad , \quad \tilde{D} = -2\,\xi' D - \xi\, D' - \frac{c\bar{\beta}}{8}\,\xi''' \quad ,$$

$$L_\xi * \bar{G}^{\bar{Q}}_{\Psi^{\bar{Q}}} = \bar{G}^{\bar{Q}}_{\tilde{\Psi}^{\bar{Q}}} \quad , \quad \tilde{\Psi}^{\bar{Q}} = -\,(\tfrac{3}{2}\xi'\psi^{\bar{Q}} + \xi(\psi^{\bar{Q}})') \quad ,$$

$$L_\xi * \bar{T}^{\bar{\rm R}\bar{\rm S}}_{\tau^{\bar{\rm R}\bar{\rm S}}} = \bar{T}^{\bar{\rm R}\bar{\rm S}}_{\tilde{\tau}^{\bar{\rm R}\bar{\rm S}}} \quad , \quad \tilde{\tau}^{\bar{\rm R}\bar{\rm S}} = -\xi'\tau^{\bar{\rm R}\bar{\rm S}} - \xi\,(\tau^{\bar{\rm R}\bar{\rm S}})' \quad ,$$

$$\begin{aligned}
L_\xi * \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}_{\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}} &= \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}_{\tilde{\omega}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}} + \frac{i}{2}(i)^{[\frac{n-2}{2}]-[\frac{n}{2}]}\,\bar{R}^{[\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n-2}}_{\xi''\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}}\,\delta^{{\rm V}_{\rm n-1}]{\rm V}_{\rm n}} \quad , \\
&\quad \tilde{\omega}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}} = (\frac{n}{2}-2)\,\xi'\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}} - \xi(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})' \quad ,
\end{aligned}$$

$$\begin{aligned}
L_\xi * \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}_{\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}} &= \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}_{\tilde{\rho}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}} \quad , \\
&\quad \tilde{\rho}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}} = (\frac{\rm m}{2}-2)\,\xi'\,\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}} - \xi\,(\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}})' \quad ,
\end{aligned}$$

$$G^{\rm I}_{\chi^{\rm I}} * \bar{G}^{\bar{\rm Q}}_{\psi^{\bar{\rm Q}}} = \delta^{{\rm I}\bar{\rm Q}}\,\bar{L}_{\tilde{\xi}} + 4\bar{T}^{{\rm I}\bar{\rm Q}}_{(\chi^{\rm I}\psi^{\bar{\rm Q}})} \quad , \quad \tilde{\xi} = \frac{1}{2}\,(\psi^{\bar{Q}})'\chi^I - \frac{3}{2}\,(\chi^I)'\psi^{\bar{Q}} \quad ,$$

$$G^{\rm I}_{\chi^{\rm I}} * (\bar{L}_D, \bar{\beta}) = 4i\,\bar{G}^{\rm I}_{(-\chi^{\rm I}\,D - \bar{\beta}c(\chi^{\rm I})'')} \quad ,$$

$$G^{\rm I}_{\chi^{\rm I}} * \bar{T}^{\bar{\rm R}\,\bar{\rm S}}_{\tau^{\bar{\rm R}\,\bar{\rm S}}} = \frac{i}{2}(\bar{G}^{\rm S}_{\chi^{\rm S}}\,\delta^{\bar{\rm R}{\rm I}} - \bar{G}^{\bar{\rm R}}_{\chi^{\bar{\rm R}}}\,\delta^{{\rm I}\bar{\rm S}}) \quad , \quad \chi^{\bar{\rm R}} = \chi^{\rm S} = 2(\chi^{\rm I})'\tau^{\bar{\rm R}\,\bar{\rm S}} + \chi^{\rm I}\,(\tau^{\bar{\rm R}\,\bar{\rm S}})' \quad ,$$

$$\begin{aligned}
G^{\rm I}_{\chi^{\rm I}} * \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}_{\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}}} &= 2i(i)^{m+1}(i)^{[\frac{m+2}{2}]-[\frac{m}{2}]}\,\bar{U}^{[\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}]}_{(\chi^{\rm I}\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}})} \\
&\quad - 2i^{[\frac{m-1}{2}]-[\frac{m-2}{2}]}\,\delta^{I\,[\bar{T}_1}\,\bar{R}^{\bar{\rm T}_2\cdots\bar{\rm T}_{\rm m}]}_{((\chi^{\rm I})'\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}} - (\chi^{\rm I})(\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}})')} \\
&\quad - (i)(i)^{[\frac{m+1}{2}]-[\frac{m}{2}]}\sum_{r=1}^{m+1}(-1)^{r-1}\,\bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm r-1}\,{\rm I}\,\bar{\rm T}_{\rm r+1}\cdots\bar{\rm T}_{\rm m}}_{(\chi^{\rm I}\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_{\rm m}})} \quad ,
\end{aligned}$$

$$T^{\rm J\,K}_{t^{\rm J\,K}} * \bar{G}^{\bar{\rm Q}}_{\psi^{\bar{\rm Q}}} = -\,2(\bar{G}^{\rm K}_{(t^{\rm J\,K}\,\psi^{\bar{\rm Q}})}\delta^{\bar{\rm Q}{\rm J}} - \bar{G}^{\rm J}_{(t^{\rm J\,K}\,\psi^{\bar{\rm Q}})}\delta^{\bar{\rm Q}{\rm K}}) \quad ,$$

$$\begin{aligned}
G^{\rm I}_{\chi^{\rm I}} * \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}_{\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}} &= -\,2i^{[\frac{n-1}{2}]-[\frac{n}{2}]}\,\delta^{I\,[\bar{\rm V}_1}\,\bar{U}^{\bar{\rm V}_2\cdots\bar{\rm V}_{\rm n}]}_{((n-4)(\chi^{\rm I})'\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}} - (\chi^{\rm I})(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})')} \\
&\quad + 2(-1)^{n-1}(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]}\,\delta^{{\rm I}[\bar{\rm V}_{\rm n}}\bar{U}^{\bar{V}_1\cdots\bar{V}_n]\,{\rm K}}_{((\chi^{\rm I})'\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})^{\rm K}} \\
&\quad + (i)(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]}\sum_{r=1}^{n}\bar{U}^{[\bar{\rm V}_1\cdots\bar{\rm V}_{\rm r-1}\bar{\rm V}_{\rm r+1}\cdots\bar{\rm V}_{\rm n}}_{(\chi^{\rm I}\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})}\,\delta^{\bar{\rm V}_{\rm r}\,]\,{\rm I}} \\
&\quad + \bar{G}^{[\bar{\rm V}_2}_{(-4i\,(\chi^{\rm I})'(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})' - 2i\,(\chi^{\rm I})(\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})'')}\,\delta^{\bar{\rm V}_3\bar{\rm V}_4}\,\delta^{\bar{\rm V}_1]{\rm I}}\,\delta^{n4} \\
&\quad - 2i(-1)^n(i)^{[\frac{n}{2}]-[\frac{n-1}{2}]}\,\delta^{{\rm I}[\bar{\rm V}_{\rm n}}\,\bar{R}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n-1}]}_{(\chi^{\rm I}\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})} \quad ,
\end{aligned}$$

$$\begin{aligned}
T^{\rm J\,K}_{t^{\rm J\,K}} * \bar{U}^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}_{\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}}} &= -\sum_{r=1}^{n-1}(-1)^{n+1}(\delta^{\rm J\,[\,\bar{\rm V}_1}\,\bar{U}^{\bar{\rm V}_2\cdots\bar{\rm V}_{\rm r-1}|\,{\rm K}\,|\bar{\rm V}_{\rm r+1}\cdots]\bar{\rm V}_{\rm n}}_{(t^{\rm J\,K}\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})} \\
&\qquad\qquad - \delta^{\rm K\,[\,\bar{\rm V}_1}\,\bar{U}^{\bar{\rm V}_2\cdots\bar{\rm V}_{\rm r-1}|\,{\rm J}\,|\bar{\rm V}_{\rm r+1}\cdots]\bar{\rm V}_{\rm n}}_{(t^{\rm J\,K}\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})}) \\
&\quad + \bar{U}^{[\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n-1}]\,{\rm J}}_{(t^{\rm J\,K}\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})}\,\delta^{\bar{\rm V}_{\rm n}\,{\rm K}} - \bar{U}^{[\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n-1}]\,{\rm K}}_{(t^{\rm J\,K}\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})}\,\delta^{\bar{\rm V}_{\rm n}\,{\rm J}} \\
&\quad - i(-1)^{n-2}\,(\delta^{{\rm K}\,\bar{\rm V}_{\rm n}}\,\delta^{J\,[\bar{V}_1} - \delta^{{\rm J}\,\bar{\rm V}_{\rm n}}\,\delta^{{\rm K}\,[\bar{\rm V}_1})\bar{R}^{\bar{\rm V}_2\cdots\bar{\rm V}_{\rm n-1}]}_{((t^{\rm J\,K})'\,\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_{\rm n}})} \quad ,
\end{aligned}$$

$$
\begin{aligned}
R^{\mathrm{J_1\cdots J_p}}_{r\{J_p\}} * \bar U^{\mathrm{\bar V_1\cdots\bar V_m}}_{\omega\{V_m\}} = \; & -\frac{1}{2} i(i)^{\{[\frac{p}{2}]-[\frac{p+2}{2}]\}}\,\delta^{[\mathrm{\bar V_1\cdots\bar V_{m-2}}}_{[\mathrm{\bar J_1\cdots \bar J_p}]}\,\delta^{\mathrm{\bar V_{m-1}],\bar V_m}}\,\delta^{\mathrm{m,p+2}}\,\bar L_{(r\omega)''} \\
& + i^{\{[\frac{p}{2}]-[\frac{p+1}{2}]\}}\delta^{\mathrm{p+1,m}}\bar G^{\mathrm{\bar V_m}}_{(r\omega)'}\,\delta^{[\mathrm{\bar V_1\cdots\bar V_{m-1}}]}_{[\mathrm{J_1\cdots J_p}]} \\
& + 2i(-1)^{p+1}\delta^{\mathrm{p+3,m}}\,(i)^{[\frac{p}{2}]-[\frac{p+2}{2}]}\bar G^{[\mathrm{\bar V_1}}_{(r\omega)''}\,\delta^{\mathrm{\bar V_2\cdots\bar V_{m-2}}}_{[\mathrm{J_1\cdots J_p}]}\,\delta^{\mathrm{\bar V_{m-1},]\bar V_m}} \\
& + i(-1)^p\bar T^{\mathrm{\bar V_1[\bar V_m}}_{(r\omega)'}\delta^{\mathrm{\bar V_2\cdots\bar V_{m-1}]}}_{[\mathrm{J_1\cdots J_p}]}\delta^{p+2,m} \\
& + (-1)^{\mathrm{pm}}(i)^{\{[\frac{m}{2}]+[\frac{p}{2}]-[\frac{m+p-2}{2}]\}}\,\bar U^{\mathrm{\bar V_p+1\cdots\bar V_{m-p}}}_{(r\omega)'}\delta^{\mathrm{\bar V_1\cdots\bar V_p}}_{[\mathrm{J_1\cdots J_p}]} \quad , \\
U^{\mathrm{I_1\cdots I_q}}_{\mu\{\mathrm{I_q}\}} * \bar U^{\mathrm{\bar V_1\cdots\bar V_m}}_{\omega\{\mathrm{V_m}\}} = \; & -\delta^{\mathrm{m\,q}}\,\delta^{[\mathrm{I_1\cdots I_q}]}_{[\mathrm{\bar V_1\cdots\bar V_q}]}\bar L_{((\frac{4-q}{2})\mu\omega-(\frac{q-2}{2})\mu\omega')} - 2\,\bar T^{\mathrm{\bar V_m J_q}}_{(\omega\mu)}\delta^{[\mathrm{I_1\cdots I_q}]}_{[\mathrm{\bar V_1\cdots\bar V_q}]}\,\delta^{\mathrm{q}}_{\mathrm{m}} \\
& - 2i^{([\frac{\bar V_1}{2}]-[\frac{\bar V_1+1}{2}])}\,\delta^{\mathrm{m,(q+1)}}\bar G^{[\mathrm{\bar V_1}}_{(-(q-2)\omega'\mu+(3-q)\omega\mu')}\,\delta^{\mathrm{\bar V_2\cdots\bar V_m}]}_{[\mathrm{I_1\cdots I_q}]} \\
& + 2(-1)^q(i)^{[\frac{q}{2}]-[\frac{q+1}{2}]}\bar G^{\mathrm{I_q}}_{(-\omega\mu)'}\,\delta^{\mathrm{m,q+1}}\,\delta^{[\mathrm{\bar V_1\cdots\bar V_{m-2}}}_{[\mathrm{I_1\cdots I_{q-1}}]}\,\delta^{\mathrm{\bar V_{m-1}],\bar V_m}} \\
& - i(i)^{[\frac{q}{2}]-[\frac{q-1}{2}]}\sum_{r=1}^{q-1}(-1)^{r-1}\delta^{\mathrm{m}}_{\mathrm{q-1}}\,\bar G^{[\mathrm{I_r}}_{(\omega\mu)}\delta^{\mathrm{I_1}}_{[\mathrm{V_1}}\cdots\delta^{\mathrm{I_{r-1}}}_{\mathrm{\bar V_{r-1}}}\,\delta^{\mathrm{I_{r+1}}}_{\mathrm{\bar V_r}}\cdots\delta^{\mathrm{I_q}]}_{\mathrm{\bar V_m}]} \\
& + \sum_{r=1}^{q-1}2(-1)^{r+1}(\bar T^{\mathrm{J_r[\bar V_1}}_{(\omega\mu)}\delta^{\bar V_2}_{[\mathrm{I_1}}\cdots\delta^{\bar V_r}_{\mathrm{I_{r-1}}}\delta^{\bar V_{r+1}}_{\mathrm{I_{r+1}}}\cdots\delta^{]\bar V_m}_{]\mathrm{I_q}}\delta^{\mathrm{q,m}}) \\
& + i(i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}+2]-[\frac{m+2}{2}]\}} \\
& \quad \times\{\sum_{r=1}^{q}(-1)^{r-1}\delta^{[\mathrm{\bar V_1\cdots\bar V_{q-1}}}_{[\mathrm{I_1\cdots I_{q-1}}]}\;\bar U^{\mathrm{\bar V_q\cdots\bar V_{q+r-1}I_q\bar V_{q+r}\cdots]V_m}}_{\omega\mu} \\
& - (-1)^{q(m-q+2)}\sum_{r=1}^{q}(-1)^{r-1}\bar U^{\mathrm{\bar V_1\cdots\bar V_{m-q+1}[I_r}}_{\omega\mu}\delta^{\mathrm{I_1\cdots I_{r-1}I_{r+1}\cdots I_q]}}_{\mathrm{\bar V_{m-q+2}\cdots\bar V_{m-q+2+r}\cdots\bar V_m}}\;\} \\
& - (i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}]-[\frac{q+m-4}{2}]\}}\;\bar R^{[\mathrm{\bar V_1\cdots\bar V_{m-q}}}_{(\omega\mu)'}\,\delta^{\mathrm{\bar V_{m-q+1}\cdots]\bar V_m}}_{[\mathrm{I_1\cdots]I_q}} \quad , \\
R^{\mathrm{J_1\cdots J_p}}_{r\{J_p\}} * \bar R^{\mathrm{\bar T_1\cdots\bar T_m}}_{\rho\{T_m\}} = \; & \delta^{[\mathrm{\bar T_1\cdots\bar T_m}]}_{[\mathrm{J_1\cdots J_p}]}\,\delta^{\mathrm{p\,m}}\,\bar L_{(-(\frac{p}{2}-2)r'\rho-(\frac{p}{2}-1)\,r\rho')} \\
& + (-1)^p\,\{2(i)\bar G^{[\mathrm{\bar T_1}}_{((2-p)r'\rho-(p-1)r\rho')}\,\delta^{\mathrm{\bar T_2\cdots\bar T_m}]}_{[\mathrm{J_1\cdots J_p}]}\delta^m_{p+1} \\
& + (i)(i)^{[\frac{p}{2}]-[\frac{p-1}{2}]}\sum_{r=1}^{p}(-1)^r\bar G^{\mathrm{J_r}}_{(r\rho)}\delta^{\mathrm{m}}_{\mathrm{p-1}}\delta^{[\mathrm{J_1\cdots J_{r-1}J_{r+1}\cdots J_p}]}_{[\mathrm{\bar T_1\cdots\bar T_{r-1}\bar T_r\ldots\bar T_m}]}\} \\
& + \sum_{r=1}^{p}(-1)^{r+1}2\bar T^{[\mathrm{\bar T_1}\,|\mathrm{J_r}|}_{(r\rho)}\delta^{\mathrm{\bar T_2\cdots\bar T_m}]}_{[\mathrm{J_1\cdots J_{r-1}\,J_{r+1}\cdots J_p}]}\delta^p_m \\
& - \sum_{r=1}^{p}(-1)^{r-1}\bar U^{[\mathrm{\bar T_1\cdots\bar T_{m-p+1}}\,|\mathrm{J_r}|}_{(r\rho)}\delta^{\mathrm{\bar T_{m-p+2}\cdots\bar T_m}]}_{\mathrm{J_1\cdots J_{r-1}J_{r+1}\cdots J_p}} \\
& + i^{\{[\frac{p}{2}]+[\frac{m-p}{2}]-[\frac{m}{2}]\}}\bar R^{[\mathrm{\bar T_{p+1}\cdots\bar T_m}}_{(2r'\rho+r\rho')}\delta^{\mathrm{\bar T_1\cdots\bar T_p}]}_{[\mathrm{J_1\cdots J_p}]} \quad ,
\end{aligned}
$$

$$T^{\rm J\,K}_{t^{\rm J\,K}} * \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_m}_{\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_m}} = \sum_{r=1}^{m} (-1)^{r+1} \, (\delta^{[\bar{\rm T}_1|\,{\rm J}\,|}\, \bar{R}^{\bar{\rm T}_2\cdots\bar{\rm T}_{r-1}\,|\,{\rm K}\,|\,\bar{\rm T}_{r+1}\cdots\bar{\rm T}_m]}_{(t^{\rm J\,K}\,\rho^{\bar{\rm V}_1\cdots\bar{\rm V}_n})} - \delta^{[\bar{\rm T}_1|\,{\rm K}\,|}\, \bar{R}^{\bar{\rm T}_2\cdots\bar{\rm T}_{r-1}\,|\,{\rm J}\,|\,\bar{\rm T}_{r+1}\cdots\bar{\rm T}_m]}_{(t^{\rm J\,K}\,\rho^{\bar{\rm V}_1\cdots\bar{\rm V}_n})}) \quad ,$$

$$U^{\rm I_1\cdots I_q}_{\mu^{\{\rm I_q\}}} * \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_m}_{\rho^{\{\rm T_m\}}} = -\, i(-1)^{q(m-q+2)}(i)^{\{[\frac{m-q}{2}+2]+[\frac{q}{2}]-[\frac{m}{2}]\}} \times \sum_{r=1}^{m-q+2} (-1)^{r-1} \delta^{[\rm I_1\cdots I_{q-1}]}_{[\bar{\rm T}_1\cdots\bar{\rm T}_{q-1}]} \bar{R}^{\bar{\rm T}_1\cdots\bar{\rm T}_{q+r-1}\,{\rm I_q}\,\bar{\rm T}_{q+r+1}\cdots\bar{\rm T}_m}_{\rho\mu} \quad ,$$

$$T^{\rm J\,K}_{t^{\rm J\,K}} * (\bar{T}^{\bar{\rm R}\bar{\rm S}}_{\tau^{\bar{\rm R}\bar{\rm S}}}, \bar{\beta}) = \frac{1}{2}(\delta^{\bar{\rm R}\rm J}\delta^{\bar{\rm S}\rm K} - \delta^{\bar{\rm R}\rm K}\delta^{\bar{\rm S}\rm J})\, \bar{L}_{((t^{\rm JK})'\,\tau^{\bar{\rm R}\,\bar{\rm S}})} + \frac{1}{2}\bar{T}^{\rm A\,B}_{(t^{\rm J\,K}\,\tau^{\bar{\rm R}\bar{\rm S}})}\, \delta^{\rm JK\bar{R}\bar{S}}_{\rm AB} + 4\bar{\beta}\bar{T}^{\rm JK}_{(\tau^{\bar{\rm R}\bar{\rm S}})'} \quad ,$$

$$\text{where} \quad \delta^{\rm JK\bar{R}\bar{S}}_{\rm AB} \equiv (\, \delta^{\rm AK}\delta^{\rm B\bar{S}}\delta^{\rm \bar{R}J} - \delta^{\rm AK}\delta^{\rm B\bar{R}}\delta^{\rm \bar{S}J} + \delta^{\rm A\bar{S}}\delta^{\rm BJ}\delta^{\rm \bar{R}K} - \delta^{\rm A\bar{R}}\delta^{\rm JS}\delta^{\rm \bar{S}K} + \delta^{\rm A\bar{S}}\delta^{\rm BK}\delta^{\rm \bar{R}J} - \delta^{\rm A\bar{R}}\delta^{\rm KB}\delta^{\rm \bar{S}J} + \delta^{\rm AJ}\delta^{\rm B\bar{S}}\delta^{\rm \bar{R}K} - \delta^{\rm JA}\delta^{\rm \bar{R}B}\delta^{\rm \bar{S}K} + \delta^{\rm A\bar{S}}\delta^{\rm BK}\delta^{\rm \bar{R}J}\,) \quad ,$$

where the symmetry of the indices on the left hand side should be imposed on the indices on the right side. In the above, we have sometimes suppressed the indices associated with the functions used by the generators. For example $\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_n}$ the associated with the $\bar{U}$ dual element may be written as $\omega^{\{\rm V_m\}}$ or simply as $\omega$. Also the notation $\delta^{\rm I_1\cdots I_m}_{\rm J_1\cdots J_m} \equiv \delta^{\rm I_1}_{\rm J_1}\cdots\delta^{\rm I_m}_{\rm J_m}$ was utilized.

## 1D Supergravity Pointing Toward D-dimensional SG

The coadjoint of the Virasoro algebra, i.e. the action of L on the coadjoint vectors reveals a spectrum of states containing:

- $D$ corresponds to a rank 2 covariant tensor when the central extension is set to zero and is otherwise a quadratic differential.
- $\psi^{\bar{\rm I}}$ corresponds to $N$ spin-$\frac{3}{2}$ fields that partner with $D$.
- $\tau^{\bar{\rm R}\bar{\rm S}}$ corresponds to the spin-1 covariant tensors that serves as the $N(N-1)/2$ SO($N$) gauge potentials associated with the supersymmetries.
- Given the $N$ supersymmetries there are the fields $\omega^{\bar{\rm V}_1\cdots\bar{\rm V}_p}$. For a fixed value of $N$, the total number of independent components is given by

$$\#(U) \;=\; N\,(\,2^N \,-\, N-1\,) \quad .$$

- Again given $N$ supersymmetries, there are the fields $\rho^{\bar{\rm T}_1\cdots\bar{\rm T}_p}$. For a fixed value of $N$, the total number of independent components is given by

$$\#(R) \;=\; (\,2^N \,-\, N-1\,) \quad .$$

| Table 2: Tensors Associated with the Dual of the Algebra | | |
|---|---|---|
| Dual element of algebra | Transformation Rule | Tensor Structure |
| ${L^\star}_{\mathcal{A}} \; \to \; h$ | $h \; \to \; - \xi h' - 2\xi' h$ | $h_{ab}$ |
| ${G^{\star \rm I}}_{\mathcal{A}} \; \to \; \psi^{\rm I}$ | $\psi^{\rm I} \; \to \; -\xi(\psi^{\rm I})' - \frac{3}{2}\xi'\psi^{\rm I}$ | $\psi^{\rm I}_{a\alpha}$ |
| $T^{\star {\rm RS}} \; \to \; A^{\rm RS}$ | $A^{\rm RS} \; \to \; - \xi \, (A^{\rm RS})' - (\xi)' A^{\rm RS}$ | $A^{\rm RS}_a$ |
| $U^{\star {\rm V_1 \cdots V_n}} \; \to \; \omega^{\rm V_1 \cdots V_n}$ | $\omega^{\rm V_1 \cdots V_n} \; \to \; - \xi(\omega^{\rm V_1 \cdots V_n})' - (2 - \frac{n}{2})\xi'\omega^{\rm V_1 \cdots V_n}$ | $\omega_{ab}^{\rm V_1 \cdots V_n ;\, \alpha_1 \cdots \alpha_n}$ |
| ${R^{\star {\rm T_1 \cdots T_r}}}_{\mathcal{A}} \; \to \; \rho^{\rm T_1 \cdots T_r}$ | $\rho^{\rm T_1 \cdots T_r} \; \to \; - \xi(\rho^{\rm T_1 \cdots T_r})' - (2 - \frac{r}{2})\xi'\rho^{\rm T_1 \cdots T_r}$ | $\rho_{ab}^{\rm T_1 \cdots T_r ; \alpha_1 \cdots \alpha_r}$ |

The spins of the fields associated with $U$ and $R$ vary according to $(2 - \frac{p}{2})$. These likely correspond to other gauge and non-gauge physical fields, auxiliary, and Stueckelberg fields that are required to close the supersymmetry algebra. The fact that the spin $S$ satifies

$$S \; = \; \tfrac{1}{2} \, ( \; 4 \; - \; p \; )$$

follows as a consequence of the super-Virasoro algebra.