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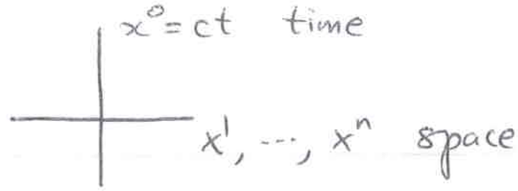
Introduction to Supersymmetry

- M^n Minkowski spacetime
  V Lorentz metric, $ds^2 = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2$



- Symmetries

$$\begin{array}{ccccccccc} 1 & \to & V & \to & \text{Symmetries} & \to & O(V) & \to & 1 \\ & & & & & & \uparrow & & \\ 1 & \to & V & \to & P^n & \to & \mathrm{Spin}(V) & \to & 1 \end{array}$$

Poincaré group

- <u>Field</u> Scalar $\phi : M^n \to \mathbb{R}$
  Wave equation $\Box\phi + m^2\phi = 0$.

Classical Lagrangian

$\mathcal{M} \subset \mathcal{F} = \Omega^0(M^n)$

$S : \mathcal{F} \to \mathbb{R}$ action

$\mathcal{M} = \{\phi : dS = 0\}$

Here: $S(\phi) = \int_M \frac{1}{2}|d\phi|^2 - \frac{m^2}{2}\phi^2$

$\xrightarrow{\text{Feynman}}$ Correlation function

$\downarrow$

$\mathcal{M}$ = solution of $\Box\phi + m^2\phi = 0$

- ∞-dim
- real repn of P^n
- symplectic vector space

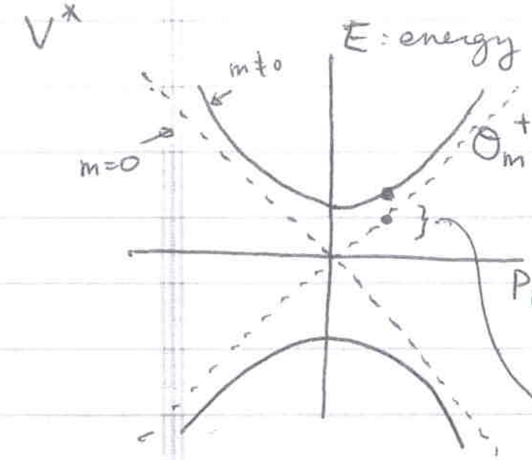
$\longrightarrow$ <u>Quantum picture</u>

$\mathcal{H} = \mathrm{Sym}^\bullet(H_1)$

(H_1 as below)

Fourier Transform

$$\hat{\phi} : V^* \to \mathbb{C}$$



$\mathcal{M}$ = $\mathbb{C}$-functions on $\mathcal{O}_m$ satisfying $\hat{\phi}(-\xi) = \overline{\hat{\phi}(\xi)}$

H_1 = $\mathbb{C}$-functions on $\mathcal{O}_m^+$

time $\leftrightarrow$ frequency (energy)

space $\leftrightarrow$ wavenumber (momentum)

Reductive stab gp $= \begin{cases} Spin_{n-1} & m^2 > 0 \\ Spin_{n-2} & m = 0 \end{cases}$.

$\parallel$

Little group L_n

— Vector repn of L_n : gauge field $\dfrac{\Omega^1(M)}{d\Omega^0(M)}$

electromag fields Maxwell eqn	Vector boson.

— Traceless Sym^2(Vector) : symmetric tensor field

	graviton.

Repn of $\left\{\begin{matrix} Spin_{n-1} \\ Spin_{n-2} \end{matrix}\right\}$ $\to$ divided by whether $\varepsilon = \pm 1$
$\to$ spin statistics

$\varepsilon = +1$: integer spin

$\varepsilon = -1$: half integer spin

One ptcl Hilbert space $\quad H_1 = \underset{\substack{\text{bosons} \\ \varepsilon = 1}}{H_1^+} \oplus \underset{\substack{\text{fermions} \\ \varepsilon = -1}}{H_1^-}$

S <u>real</u> spin repn of $Spin(V)$

$\mathcal{F}_f : \{ \psi : M^n \longrightarrow \underset{\text{odd vector space}}{\Pi S} \}$

$$\Pi S = \operatorname{Spec} Sym^{\bullet}(\underset{\text{odd}}{S^*})$$ $\rightsquigarrow$ exterior algebra

General picture:

$$\begin{array}{cl} \mathcal{F}_f & \text{odd vector bundle} \\ & \infty\text{-dim fiber} \\ \downarrow & \\ \mathcal{F}_b & \text{bosonic fields} \\ & \infty\text{-dim manifold} \end{array}$$

n	$Spin(V)$	$\dim S$ (real) minimal
1	$\mathbb{Z}/2\mathbb{Z}$	1
2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^{>0}$	1
3	$SL_2\mathbb{R}$	2
4	$SL_2\mathbb{C}$	4
5	$Sp(1,1)$	8
6	$SL_2\mathbb{H}$	8
7		16
8		16
9		16
10	"$SL_2\mathbb{O}$"	16
11		32

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① $n \geq 1$ $\quad \mathfrak{p}^n = V \rtimes \mathfrak{so}_n$

② S real spin repn of $Spin(1, n-1)$

$s = \dim S$

$$\mathfrak{p}^{n|s} = \underset{\text{even}}{\mathfrak{p}^n} \oplus \underset{\text{odd}}{S^*} \qquad \mathbb{Z}_2\text{-graded Lie alg}$$

brakets : $\left[\mathfrak{so}_n, \begin{matrix} V \\ S^* \end{matrix} \right] \to$ representation of $\mathfrak{so}_n$ on $\left\{ \begin{matrix} V \\ S^* \end{matrix} \right\}$

$$[V, S^*] = 0$$

$$[S^*, S^*] \subset V$$

$$\Gamma : S^* \otimes S^* \to V \qquad \text{Symmetric pairing}$$

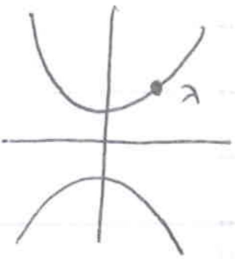
$$\tilde{\Gamma} : S \otimes S \to V.$$

$\Gamma, \tilde{\Gamma}$ related by Clifford identity

H_1 repn of $P^{n|s}$ → restrict to P^n

Little group : $\mathrm{Spin}_{n-1} \ltimes \Pi S^*$

Spin_{n-2}



$$q : S^* \otimes S^* \to V \xrightarrow{\lambda} \mathbb{R}$$

q definite for $m^2 > 0$
semidefinite for $m = 0$

Example : $P^{10|16}$

$\underline{m=0}$ Little group : $\mathrm{Spin}_8 \ltimes \mathrm{Cliff}_8$

$\cong \mathrm{Spin}(W) \ltimes \mathrm{Cliff}(\$^+)$

Minimal $\mathbb{C}$-repn : $W \oplus \$^-$
even (vector ptcl) odd (spinor)

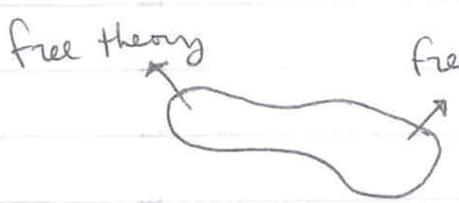
8-dim real
W
$\$^+$
$\$^-$

Another repn : $W \otimes (W \oplus \$^-)$

$W \otimes W$ bosonic $\quad W \otimes \$^-$ fermionic → turns out physically interesting

$$W \otimes W \cong \mathrm{Sym}_0^2 W \oplus \Lambda^2 W \oplus \mathbb{C}$$

- Study family of QFTs. (free theory on certain limits → "duality").



T : parameter space of QFTs.

- Topological methods
- SUSY can impose some holomorphic condition on physical quantities → useful to study non-free region.