V. Kac

Representation of linearly compact Lie superalgebras and the Standard Model

Linearly compact topological space = $\prod \left(\begin{matrix} \text{finite dim spaces} \\ \text{with discrete top} \end{matrix} \right)$

1. fin-dim space with discrete top
2. $\mathbb{C}[[x_1, \dots, x_n]]$

Problem: Classify all inf-dim simple lin. compact Lie superalgebras. L.

Example $W(m|n) = \left\{ \sum_{i=1}^{m} P_i(x,\xi)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi)\frac{\partial}{\partial \xi_j} \right\}$.

Proposition. Any $L \underset{\text{closed}}{\hookrightarrow} W(m|n)$ minimal m, then minimal n

Example 2 $S(m|n) = \{ D \in W(m|n) \mid \operatorname{div} D = 0 \}$.

$H(m|n)$

$K(m|n)$. ($K(1|n)$ this morning).

$HO(m|m)$ BV algebra

...

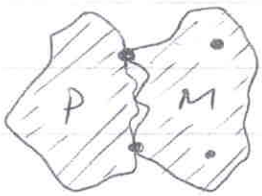
Theorem (Adv. Math 1998) A complete list of simple ∞-dim lin. compact Lie superalgebras is as follows:

I. 10 series: $W(m|n)$, $S(m|n)$, ...

II. exceptional: $E(1|6)$, $E(4|4)$, $E(3|6)$, $E(3|8)$ $E(5|10)$

E. Cartan. In the Lie algebra case, the answer is

$$W_m = W(m,0), \quad S_m, \quad H_m, \quad K_m.$$

Physics $\rightarrow$ 

$\leftarrow$ mathematics

$K = SU(3)\times SU(2)\times U(1)$

$\mathbb{Z}/6\mathbb{Z}$

$$\begin{array}{llll} & E(3|6) & \subset & E(5|10) \\ & \cup & & \\ \text{max compact} & SU(3)\times SU(2)\times U(1) & \subset & SU(5) \\ & \cap \quad \text{standard model} & & \text{Grand Unified Theory} \\ & E(3|8) \leftarrow (\not\subset E(5|10)) & & \end{array}$$

$$\boxed{E(5|10) = \underset{\text{even}}{S_5} \oplus \underset{\text{odd}}{\Omega^2_{\text{closed}}}}$$

with braket:

$$[\omega, \omega'] = \omega\wedge\omega' \in \Omega^4_{cl} \equiv S_5$$

$$i_D(dx_1\wedge\cdots dx_5) \leftarrow D$$

$E(5|10) = \bigoplus_{j=-2}^{+\infty} G_j$ defined by letting

$$\deg x_i = 2, \quad \deg \frac{\partial}{\partial x_i} = -2$$

$$\deg dx_i \wedge dx_j = -1$$

G_{-2}	G_{-1}	G_0	G_1
$= \mathbb{C}^{5*}$	$= \Lambda^2\mathbb{C}^5$	$= SL(5)$	
$\frac{\partial}{\partial x_i}$	$dx_i \wedge dx_j$	$\sum a_{ij} x_i \frac{\partial}{\partial x_j}$, $tr(a_{ij}) = 0$	45-dim

$E(3|6) \subset E(5|10)$ secondary grading:

0th piece in this grading $\rightarrow$

$$\left(\begin{array}{l} \deg x_1, x_2, x_3 = 0 \\ \deg (x_4 = z_+) = 1 \\ \deg (x_5 = z_-) = 1 \\ \deg d = -\frac{1}{2} \end{array} \right.$$

$$E(3|6) \quad \mathfrak{g}_{-2} \quad \mathfrak{g}_{-1} \quad \underset{sl(3)+sl(2)+gl(1)}{\mathfrak{g}_0} \quad \mathfrak{g}_1 \quad \mathfrak{g}_2$$

$\mathfrak{g}_0$ = centralizer of hypercharge operator

$$Y = \mathrm{diag}\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -1\right)$$

Want irreducible L-modules in linearly compact space

<u>Example</u> $L = W_n$; $\Omega^k \supset d\Omega^{k-1}$

dualizing, i.e. cont. irrep in a vec space with discrete top,

then $\Omega^{k*} = \mathrm{Ind}_{L_0}^{L} \left(\begin{array}{c} \text{finite-dim irrep of } L_0 \\ (\equiv \text{ of } \mathfrak{g}_0) \end{array} \right)$

L_0 = subalg of vector fields preserving origin.

Proposition: Any irred. cont L-module in a v. space V with discrete top. is a quotient of $\mathrm{Ind}_{L_0}^{L} U$ (where U is an irrep of $\mathfrak{g}_0$) by a max. submodule.

Def A repn $\mathrm{Ind}_{L_0}^{L} U$ is called degenerate if it is not irreducible. The irred. quotient of such is called a degenerate irrep.

Theorem (Rudakov 74) All degenerate irrep of W_m are cokernels of d^* in Ω^*.

$\mathfrak{g}_0 = sl(3) + sl(2) + \mathbb{C}Y$ We want to clarify degenerate irreps for $E(3|6)$, ...

$E(3|6)$ induced module

$M(p, q; r; y)$

$E(5|10)$ $p, q, r \in \mathbb{Z}_+$

Corresponding irrep $y \in \mathbb{C}$

$I(p, q; r; y)$

Comm. Math. Phys (2001) Rudakov, V. K.

Construction of the $E(3|6)$ - complex.

Consider the following $\mathfrak{g}_0$ - modules:

$$V_A = \mathbb{C}[x_1, x_2, x_3, z_+, z_-]$$

$$V_B = \mathbb{C}[x_1, x_2, x_3, \partial_+, \partial_-]_{[2]}$$

$$V_C = \mathbb{C}[\partial_1, \partial_2, \partial_3, z_+, z_-]_{[-2]}$$

$$V_D = \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_+, \partial_-]$$

Subscript $[a]$ means $Y \to Y + a I_V$

Bigrading

$$V_X = \bigoplus_{m,n} V_X^{m,n} \text{ by } \deg x_i = (1,0), \deg z_\pm = (0,1)$$

Let $M_X = \mathrm{Ind}_{L_0}^{L} V_X = \bigoplus_{m,n} M_X^{(m,n)}$

Complex:

$$M = \bigoplus_{(m,n) \neq (0,0)} M_A^{(m,n)} \oplus M_B \oplus M_C \oplus \bigoplus_{(m,n) \neq (0,0)} M_D^{(m,n)}$$

Differential:

$$\nabla = \sum_i u_i \otimes b_i$$

acts on $\mathrm{Ind}_{L_0}^{L} V = \mathcal{U}(L) \otimes_{\mathcal{U}(L_0)} V$

by:

$$\nabla(u \otimes v) = \sum_i u u_i \otimes v b_i \, .$$

Example of usual de Rham d^*:

$$M = \mathbb{C}\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right] \otimes \Lambda\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_m}\right)$$

$$d^* = \sum_i \frac{\partial}{\partial x_i} \otimes \xi_i$$

Notation:

$$x_1,\ x_2,\ x_3,\ z_+ = x_4,\ z_- = x_5$$

$\partial_1, \partial_2, \partial_3, \partial_+, \partial_-$ partial derivatives

$$d_i^{\pm} = dx_i \wedge dz_{\pm} \in \mathfrak{g}_{-1}$$

$$\Delta^{\pm} = \sum_{i=1}^{3} d_i^{\pm} \otimes \partial_i, \quad \delta_i = d_i^{+} \otimes \partial_+ + d_i^{-} \otimes \partial_-$$

Differentials:

$$\nabla_1 = \Delta^{+} \partial_+ + \Delta^{-} \partial_-$$

$$\nabla_2 = \Delta^{+} \Delta^{-}$$

$$\nabla_3 = \delta_1 \delta_2 \delta_3$$

$$\nabla_4' = a \Delta^{-} \partial_+^2 + b \Delta^{-} \partial_+ \partial_- + c \Delta^{-} \partial_-^2$$

$$a = d_1^{+} d_2^{+} d_3^{+},$$

$$b = d_1^{-} d_2^{+} d_3^{+} + d_1^{+} d_2^{-} d_3^{-} + d_1^{+} d_2^{+} d_3^{-}$$

$$c = d_1^{-} d_2^{-} d_3^{+} + d_1^{-} d_2^{+} d_3^{-} + d_1^{+} d_2^{-} d_3^{-}$$

$$\nabla_4'' = d_1^{-} A \partial_1 + d_2^{-} A \partial_2 + d_3^{-} A \partial_3$$

$$A = a \partial_+^2 + b \partial_+ \partial_- + c \partial_-^2$$

$E(3,6)$

$y_C = -\frac{2}{3}q - r - 2$ $\qquad$ $y_A = \frac{2}{3}p - r$

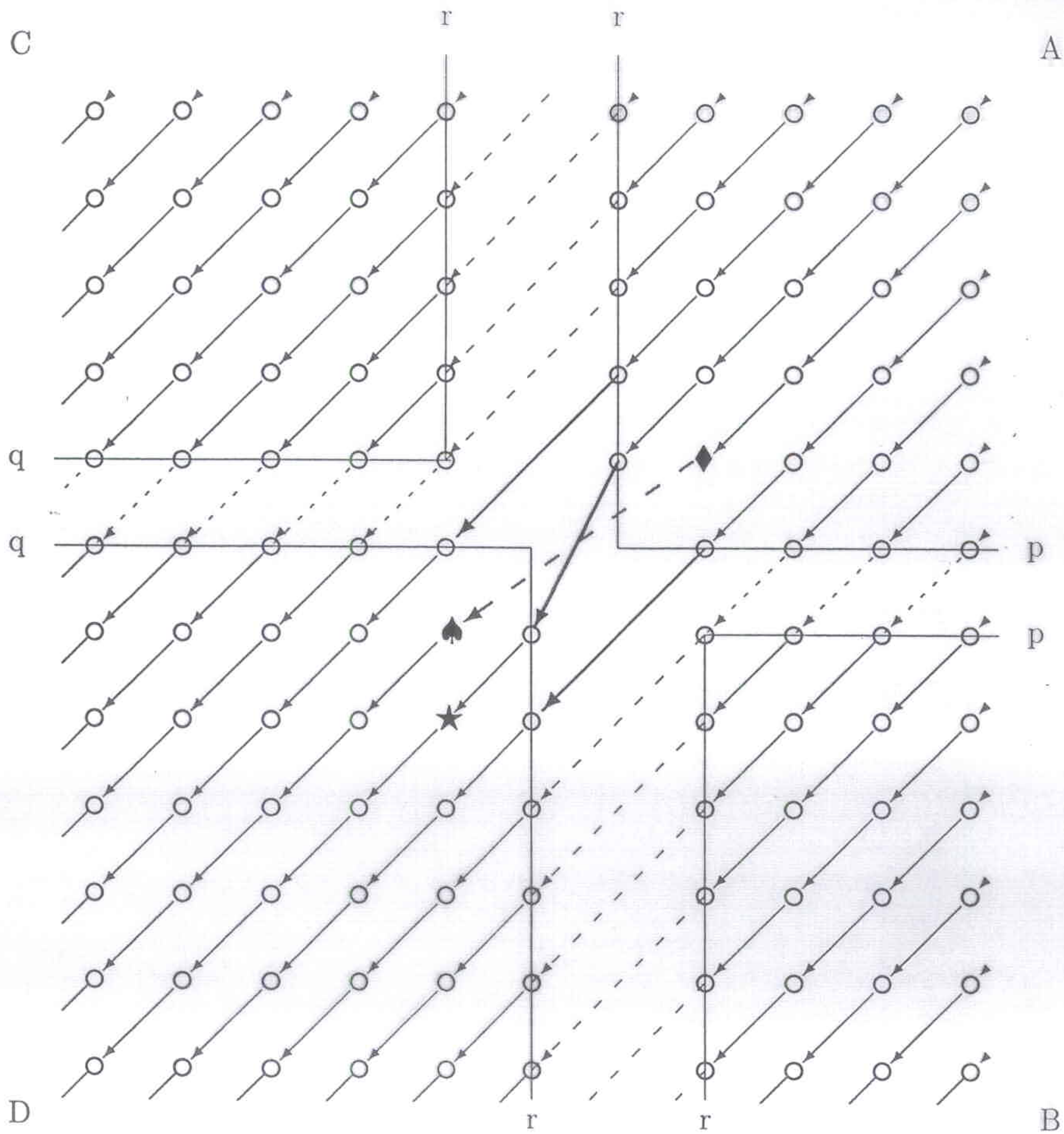


FIGURE 3

$y_D = -\frac{2}{3}q + r$ $\qquad$ $y_B = \frac{2}{3}p + r + 2$

$\star$ $H = \mathbb{C}$

♦ $H = I(00, 1, (-1)\, 6)$

♠ $H = I(00, 1, -1) \oplus \mathbb{C}$

VK, Rudakov CMP 2001

$y_C = -\frac{2}{3}q - r - 2$ $\qquad$ $y_A = \frac{2}{3}p - r$

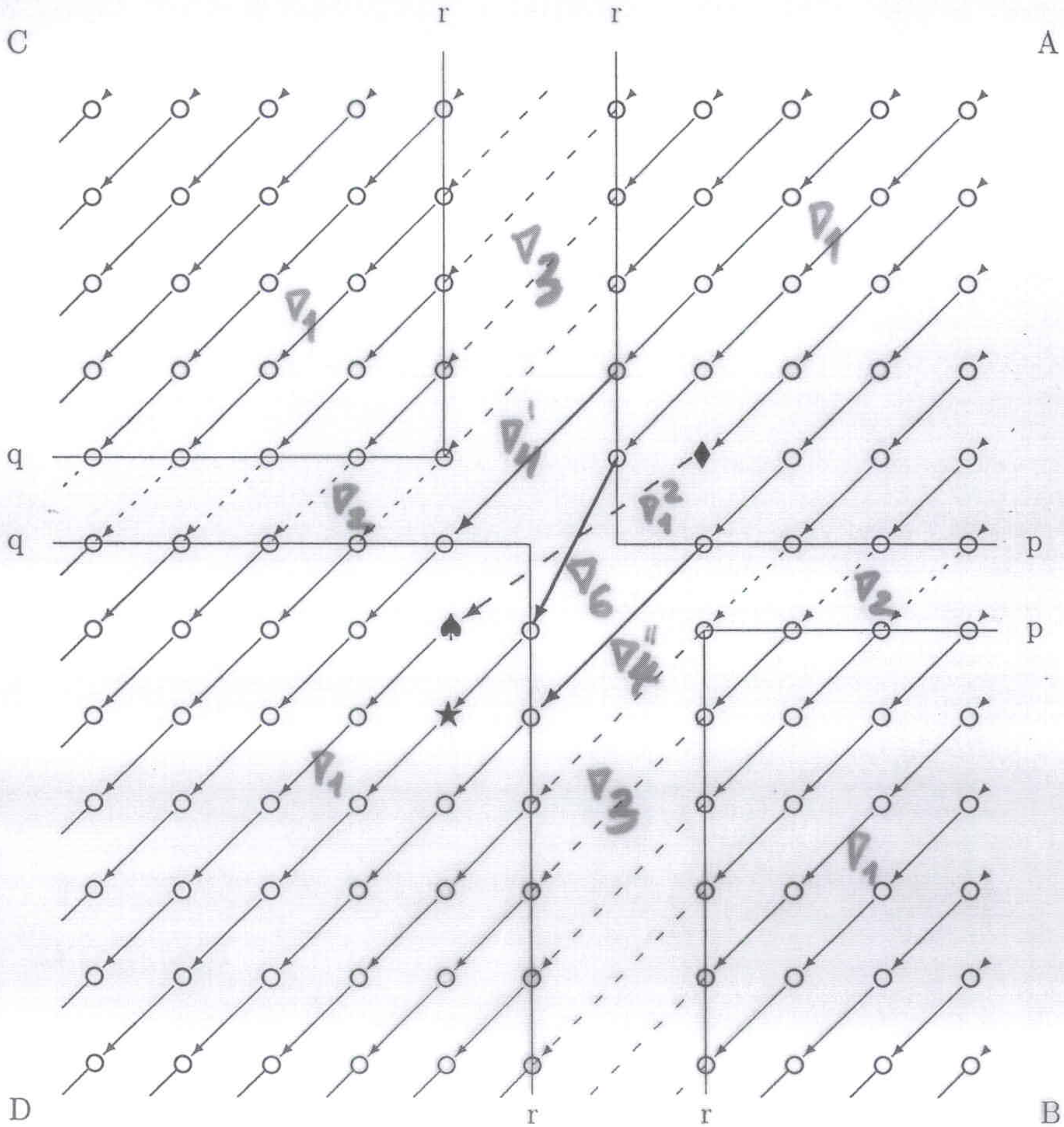


FIGURE 3

$y_D = -\frac{2}{3}q + r$ $\qquad$ $y_B = \frac{2}{3}p + r + 2$

Theorem 3. The complete list of fundamental particle multiplets is as follows:

	multiplets	charges	particles 1st	2nd	3d
quarks	$(01, 1, \frac{1}{3})$	$\frac{2}{3}, -\frac{1}{3}$	$\binom{u_L}{d_L}$	$\binom{c_L}{s_L}$	$\binom{t_L}{b_L}$
	$(10, 1, -\frac{1}{3})$	$-\frac{2}{3}, \frac{1}{3}$	$\binom{\tilde{u}_R}{\tilde{u}_L}$	$\binom{\tilde{c}_R}{\tilde{s}_R}$	$\binom{\tilde{t}_R}{\tilde{b}_R}$
	$(10, 0, -\frac{4}{3})$	$-\frac{2}{3}$	$\tilde{u}_L$	$\tilde{c}_L$	$\tilde{t}_L$
	$(01, 0, \frac{4}{3})$	$\frac{2}{3}$	u_R	c_R	t_R
	$(01, 0, -\frac{2}{3})$	$-\frac{1}{3}$	d_R	s_R	b_R
	$(10, 0, \frac{2}{3})$	$\frac{1}{3}$	$\tilde{d}_L$	$\tilde{s}_L$	$\tilde{b}_L$
leptons	$(00, 1, -1)$	$0, -1$	$\binom{\nu_L}{e_L}$	$\binom{\nu_{\mu L}}{\mu_L}$	$\binom{\nu_{\tau L}}{\tau_L}$
	$(00, 1, 1)$	$0, 1$	$\binom{\tilde{\nu}_R}{\tilde{e}_R}$	$\binom{\tilde{\nu}_{\mu R}}{\tilde{\mu}_R}$	$\binom{\tilde{\nu}_{\tau R}}{\tilde{\tau}_R}$
	$(00, 0, 2)$	1	$\tilde{e}_L$	$\tilde{\mu}_L$	$\tilde{\tau}_L$
	$(00, 0, -2)$	-1	e_R	μ_R	τ_R
bosons	$(11, 0, 0)$	0	gluons		
	$(00, 2, 0)$	$1, -1, 0$	W^+, W^-, Z		
	$(00, 0, 0)$	0	γ	photon	

E(3, 6)

$y_C = -\frac{2}{3}q - r - 2$

$y_A = \frac{2}{3}p - r$

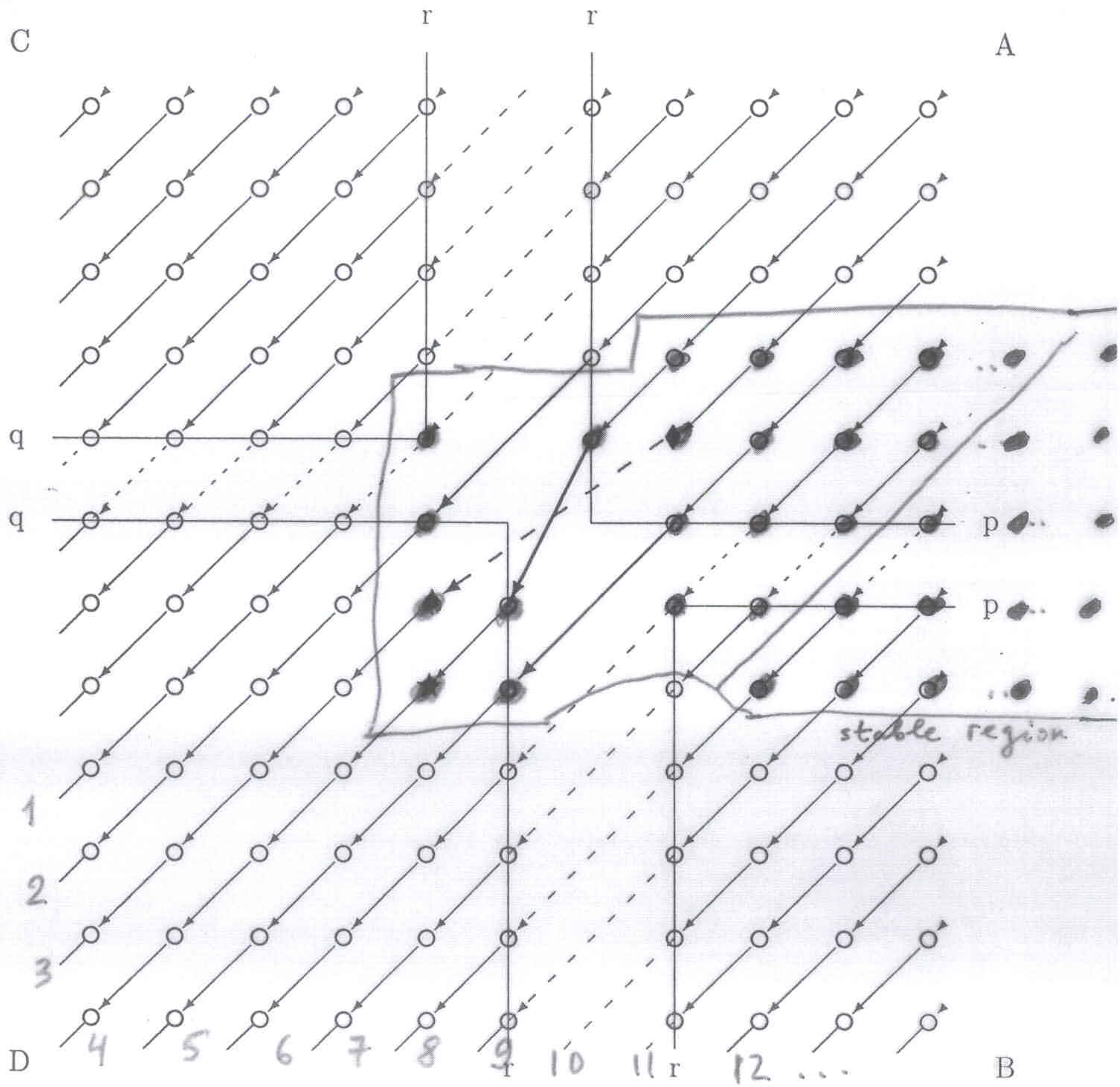


FIGURE 3

$y_D = -\frac{2}{3}q + r$

$y_B = \frac{2}{3}p + r + 2$

Sequence 1:	$I(00,0,-2)$
	$(00,0,-2)$

Sequence 2:	$I(01,0,-\frac{2}{3})$
	$2(10,0,-\frac{4}{3})$
	$(01,0,-\frac{2}{3})$
	$(00,0,-2)$
	$(00,1,-1)$

Sequence 3:	$I(01,1,\frac{1}{3})$	$I(10,1,-\frac{1}{3})$	$I(20,2,-\frac{2}{3})$
	$2(10,0,-\frac{4}{3})$	$(10,0,-\frac{4}{3})$	$(10,0,-\frac{4}{3})$
	$2(01,0,-\frac{2}{3})$	$(01,0,-\frac{2}{3})$	$(00,0,-2)$
	$2(10,1,-\frac{1}{3})$	$(10,1,-\frac{1}{3})$	
	$(01,1,\frac{1}{3})$	$2(00,0,-2)$	
	$(00,1,-1)$	$(00,1,-1)$	
	$(11,0,0)$		
	$(00,2,0)$		

Sequence 4:	$I(01,2,\frac{4}{3})$	$I(00,1,1)$	$I(00,1,-1)$	$I(10,2,-\frac{4}{3})$
	$2(10,1,-\frac{1}{3})$	$3(10,1,-\frac{1}{3})$	$(10,0,-\frac{4}{3})$	$(00,0,-2)$
	$2(01,1,\frac{1}{3})$	$2(01,1,\frac{1}{3})$	$(00,1,-1)$	
	$(11,0,0)$	$(10,0,-\frac{4}{3})$		
	$(00,2,0)$	$(10,0,\frac{2}{3})$		
		$3(01,0,-\frac{2}{3})$		
		$2(00,1,-1)$		
		$\cancel{2}(00,1,1)$		
		$2(11,0,0)$		
		$2(00,2,0)$		
		$(00,0,0)$		

Sequence 5:	$I(00,2,2)$	$I(10,0,\frac{2}{3})$	$I(20,1,\frac{1}{3})$	$I(30,2,0)$	
	$(10,1,-\frac{1}{3})$	$(10,1,-\frac{1}{3})$	$2(10,1,-\frac{1}{3})$	$(10,0,-\frac{4}{3})$	
	$3(01,1,\frac{1}{3})$	$(01,1,\frac{1}{3})$	$4(10,0,-\frac{4}{3})$	$(00,0,-2)$	
	$(01,0,\frac{4}{3})$	$(01,0,-\frac{2}{3})$	$2(01,0,-\frac{2}{3})$		
	$(10,0,\frac{2}{3})$	$(10,0,\frac{2}{3})$	$3(00,1,-1)$		
	$2(00,1,1)$	$(11,0,0)$	$2(00,0,-2)$		
	$(11,0,0)$	$(00,2,0)$	$(11,0,0)$		
	$2(00,2,0)$	$(00,0,0)$			
Sequence 6:	$I(00,0,2)$	$I(20,0,\frac{4}{3})$	$I(30,1,1)$	$I(40,2,\frac{2}{3})$	
	$(01,1,\frac{1}{3})$	$2(01,1,\frac{1}{3})$	$3(10,1,-\frac{1}{3})$	$(10,0,-\frac{4}{3})$	
	$(01,0,\frac{4}{3})$	$2(10,1,-\frac{1}{3})$	$4(10,0,-\frac{4}{3})$	$(00,0,-2)$	
	$(00,1,1)$	$2(01,0,-\frac{2}{3})$	$2(01,0,-\frac{2}{3})$		
	$(00,0,2)$	$2(10,0,\frac{2}{3})$	$3(00,1,-1)$		
		$2(11,0,0)$	$2(00,0,-2)$		
		$(00,2,0)$	$2(11,0,0)$		
		$2(00,0,0)$			
Sequence 7:	$I(10,0,\frac{8}{3})$	$I(30,0,2)$	$I(40,1,\frac{5}{3})$	$I(50,2,\frac{4}{3})$	
	$3(01,1,\frac{1}{3})$	$2(01,1,-\frac{1}{3})$	$3(10,1,-\frac{1}{3})$	$(10,0,-\frac{4}{3})$	
	$3(01,0,\frac{4}{3})$	$2(10,1,-\frac{1}{3})$	$4(10,0,-\frac{4}{3})$	$(00,0,-2)$	
	$2(10,0,\frac{2}{3})$	$2(01,0,-\frac{2}{3})$	$2(01,0,-\frac{2}{3})$		
	$3(00,1,1)$	$2(10,0,\frac{2}{3})$	$3(00,1,-1)$		
	$(00,0,2)$	$3(11,0,0)$	$2(00,0,-2)$		
	$2(11,0,0)$	$(00,2,0))$	$2(11,0,0)$		
		$2(00,0,0)$			
Sequence s≥8:	$I(s-7,0;1;\frac{2s-5}{3})$	$I(s-6,0;0;\frac{2s-6}{3})$	$I(s-4,0;0;\frac{2s-8}{3})$	$I(s-3,0;1;\frac{2s-9}{3})$	$I(s-2,0;2;\frac{2s-10}{3})$
	$(01,0,\frac{4}{3})$	$3(01,1,\frac{1}{3})$	$2(01,1,\frac{1}{3})$	$3(10,1,-\frac{1}{3})$	$(10,0,-\frac{4}{3})$
	$(00,0,2)$	$4(01,0,\frac{4}{3})$	$2(10,1,-\frac{1}{3})$	$4(10,0,-\frac{4}{3})$	$(00,0,-2)$
		$2(10,0,\frac{2}{3})$	$2(01,0,-\frac{2}{3})$	$2(01,0,-\frac{2}{3})$	
		$3(00,1,1)$	$2(10,0,\frac{2}{3})$	$3(00,1,-1)$	
		$2(00,0,2)$	$3(11,0,0)$	$2(00,0,-2)$	
		$2(11,0,0)$	$(00,2,0)$	$2(11,0,0)$	
			$2(00,0,0)$		

CPT symmetic !

The early history of the World

particles \ time	1	2	3	4	5	6	7	≥ 8
e_R	1	1	3	1	3	3	3	3
$\tilde{e}_L$	0	0	0	0	0	1	1	3
$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	0	1	2	3	3	3	3	3
$\begin{pmatrix} \tilde{\nu}_R \\ \tilde{e}_R \end{pmatrix}$	0	0	0	1	2	1	3	3
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	0	0	1	4	5	3	5	5
$\begin{pmatrix} \tilde{u}_R \\ \tilde{d}_R \end{pmatrix}$	0	0	3	5	4	5	5	5
$\tilde{u}_L$	0	2	4	2	5	5	5	5
u_R	0	0	0	0	1	1	3	5
$\tilde{d}_L$	0	0	0	1	2	1	4	4
d_R	0	1	3	3	3	4	4	4
γ	0	0	0	1	1	2	2	2
W	0	0	1	3	3	1	1	1
G	0	0	1	3	3	4	7	7

$E(3,8)$

$y_C = \frac{4}{3}q + r + 4$

$y_A = -\frac{4}{3}p + r$

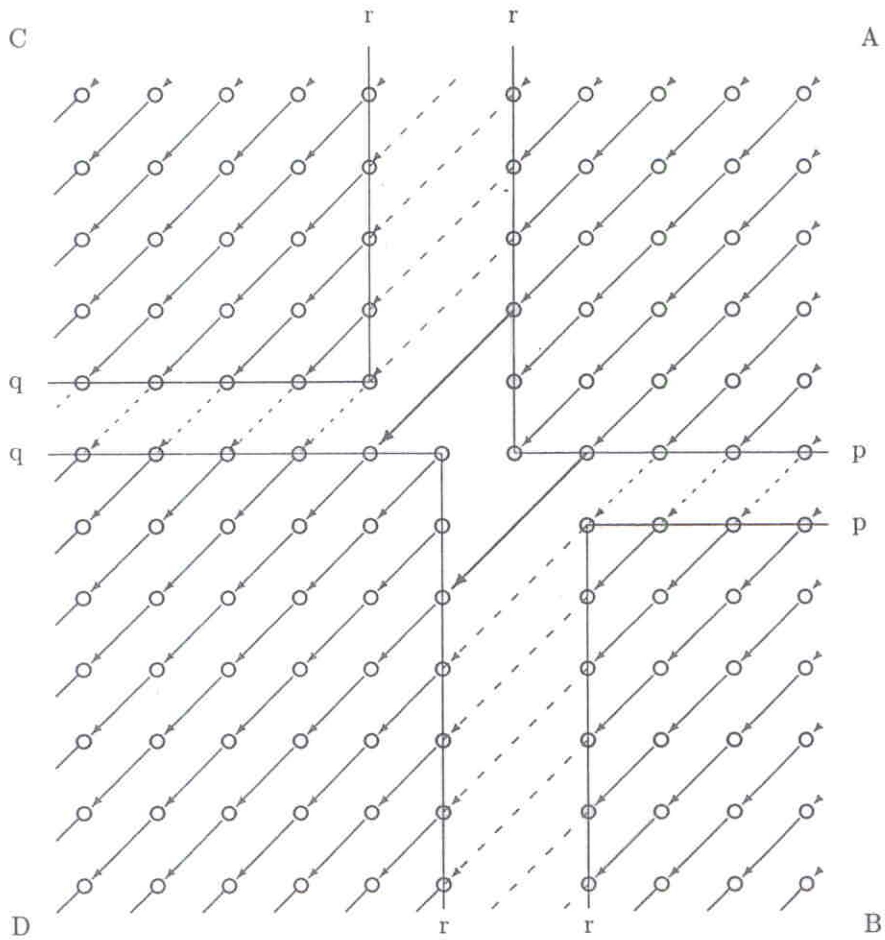


FIGURE 1

$y_D = \frac{4}{3}q - r + 2$

$y_B = -\frac{4}{3}p - r - 2$

VK - Rudakov
(IMRN, 2002)

E (3, 8)

$y_C = \frac{4}{3} q + r + 4$

$y_A = -\frac{4}{3} p + r$

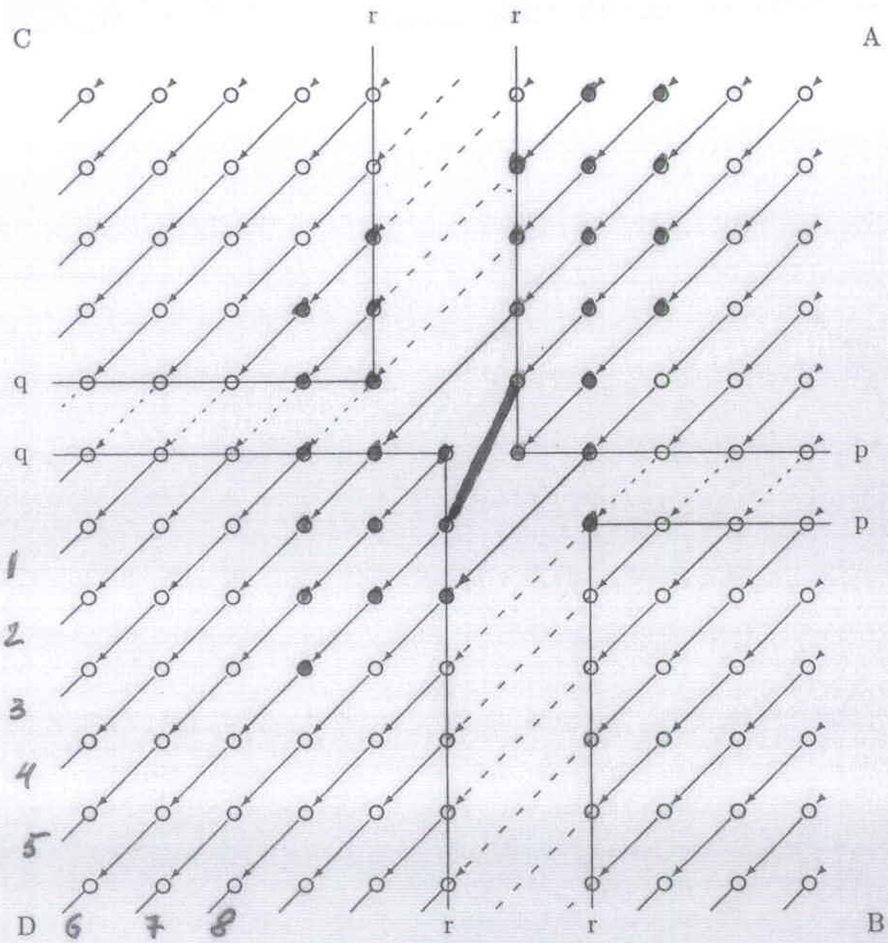


FIGURE 1

$y_D = \frac{4}{3} q - r + 2$

$y_B = -\frac{4}{3} p - r - 2$

E(5, 10)

VK - Rudakov
(IMRN, 2002)

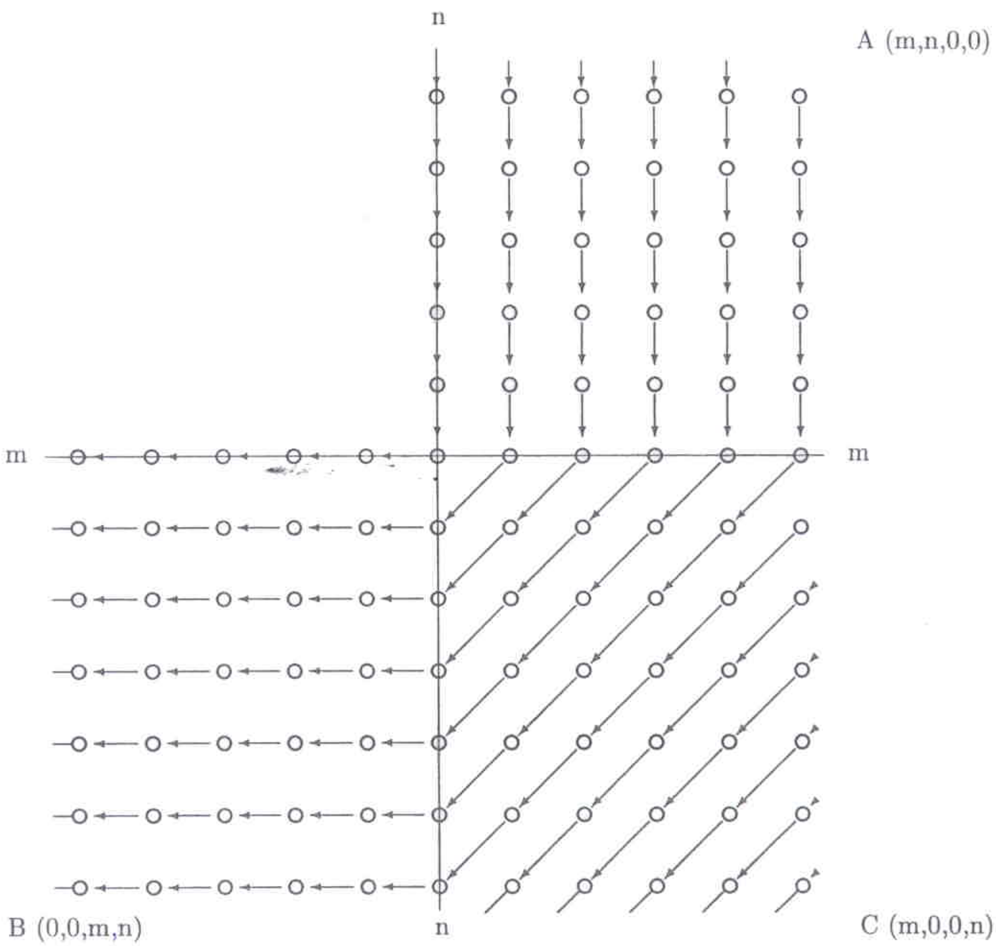


FIGURE 2