J. Morgan.

Variations of Hodge structure associated to Calabi-Yau threefolds with $h^3 = 4$.

(joint work with Chuck Doran)

- Basics of variations of Hodge str.
- CY mflds 3-folds
  $(h^{2,1} = 1 \Leftrightarrow b^3 = 4)$.
- Lemmas, Questions, Conjectures.

X proj alg // $\mathbb{C}$ smooth.

$$H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \qquad \overline{H^{p,q}} = H^{q,p}$$

Hodge filtration

$$F^r \subset H^k(X;\mathbb{C}) \qquad F^k = H^{k,0}$$

$$F^r = \bigoplus_{p \geq r} H^{p,q} \qquad F^{k-1} = H^{k,0} \oplus H^{k-1,1}$$

$$\vdots$$

$$F^p \cap \bar{F}^q = H^{p,q}$$

$$F^p \cap \bar{F}^{q+1} = 0 \quad (p+q=k)$$

primitive cohomology $P^k \subset H^k$

$$(\cup\omega)^{n-k} : H^k \xrightarrow{\cong} H^{2n-k}$$

$$P^k = \ker(\cup\omega^{n-k+1}).$$

Riemann Bilinear relations

$$\langle\ ,\ \rangle : H^k \otimes H^k \to \mathbb{Z}$$

$$a \otimes b \mapsto \langle a \cup b \cup w^{n-k}, [X] \rangle = \int_X a \wedge b \wedge w^{n-k}$$

$$\left(\begin{array}{ll} \text{Symm.} & k \equiv 0 \ (2) \\ \text{skew} & k \equiv 1 \ (2). \end{array} \right).$$

- $\langle F^p, F^{p'} \rangle = 0$ if $p + p' > k$

$\alpha \in H^{p,q}$ $\quad i^{(f(p))} \langle \alpha, \bar{\alpha} \rangle > 0$

some power depending on p.

H^3 of 3-fold.

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H^3_{\mathbb{C}}.$$

$$F^2 = (F^2)^{\perp}$$

$$F^1 = (F^3)^{\perp}$$

Fix $L, \langle\ ,\ \rangle$ $\qquad$ fix the $h^{p,q}$

on $V = L \otimes_{\mathbb{Z}} \mathbb{C}$, we consider filtrations F satisfying RB relations.

$$\mathrm{Auto}(L, \langle\ ,\ \rangle) \to T \backslash \tilde{D}.$$

$\tilde{D}$: domain of all possible filtrations.

$$\| $$

D. : all Hodge structures w/ given topological data

$$\begin{array}{l} \mathcal{X} \subset \mathbb{P}^N \\ \downarrow pr. \\ S \end{array}$$

Pick $s_0 \in S$

identify $L, \langle , \rangle$ with $H^k(X_{s_0}; \mathbb{Z})$

locally $U \to D$ which records

Hodge str. on the H^k(fibers)

Get $S \to D$ classifying map for Hodge str.

This map is holomorphic

The filtrations give a holomorphic subbundle of the flat coh. bundle.

- Griffiths horizontality

$$\nabla(\mathcal{F}^k) \subset \mathcal{F}^{k-1}$$

3 folds, $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$.

$$0 \subset \underset{\text{line}}{\underset{\|}{F^3}} \subset \underset{\text{2-plane}}{\underset{\|}{F^2}} \subset (F^3)^\perp \subset H^3_{\mathbb{C}}$$

$$(F^2)^\perp = F^2$$

$Gr(2,4)$ has dim 4

$\{F^2 \mid F^2 = F^{2\perp}\}$ ~~&~~ dim 3.

$[F^3] \in P(F^2)$ → adds 1 more dim

→ dim $D = 4$.

Horizontality:

Horiz. distribution = 2-plane distribution in D.

non-integrable

max. horizontal subspaces are curves.

Basic Question: What are all the complete horizontal curves in D?

Suppose X is a CY 3-fold

① $\underset{\text{rk } 1}{\underset{\uparrow}{H^{2,1}}} = H^1(X;\Omega^2_X) = H^1(X;TX)$ = Tangent space to deformations of X.

(Tian) Deformation space is unobstructed.

(Griffiths)

$$\nabla : \mathcal{F}^k/\mathcal{F}^{k+1} \longrightarrow \Omega^1(\mathcal{F}^{k-1}/\mathcal{F}^k)$$

$$\overset{\|}{H^{k,n-k}} \xrightarrow{\cup\tau} H^{k-1,n-k+1}$$

$$\tau \in H^1(X;TX)$$

$$H^1(X;TX) \otimes \underset{\mathbb{C}}{\underset{\| \,\Omega}{H^{3,0}}} \longrightarrow H^{2,1}.$$

$$\cong$$

$$\mathcal{M}_{CY(X)} \xrightarrow{\text{immerses}} D$$

$$\begin{array}{ccc} H^1(X; TX) \otimes H^{2,1} & \longrightarrow & H^{1,2} \\ \parallel & & \parallel \\ H^{2,1} & & (H^{2,1})^* \end{array}$$

Y = cubic form on $H^{2,1}$
describes variation of $\mathcal{F}^2$ into $\mathcal{F}^1$

Question: Is Y always nonzero? Yes, for complete intersections in toric variety.

Complete curves C in D.

$$\left(\begin{array}{c} V^4 \supset L, \overset{\text{skew}}{\langle , \rangle}, \mathcal{F}^\circ, \nabla_{G-M} \\ \downarrow \\ C \end{array} \right)$$

(Borel) The monodromies at ∞ are quasi-unipotent.

What are repns of $\pi_1(C) \to Sympl(4, \mathbb{Z})$
so that the repn at ∞ are quasi-unipotent?

Lemma Assume that one of the monod at ∞ is maximal unipotent. Then given $\pi_1(C) \to Sympl(4, \mathbb{Z})$, there is at most one variation of H.S w/ this rep. up to conj.

Assume $C = \mathbb{P}^1 - \{0, 1, \infty\}$.

<u>Mirror symmetry</u> (Homological MS of Kontsevich)

The monodromy of ∇ is isomorphic to a group of auto of $K^0(X^{dual})$

Unimodular Skew pairing
$\langle \xi, \eta \rangle = \int_{X^{dual}} ch(\xi \otimes \eta^*) \cdot Td(X^{dual})$
skew $\Leftrightarrow$ Td is even, integral by index thm unimodular by direct computation.

$h^0 = h^2 = h^4 = h^6 = 1$

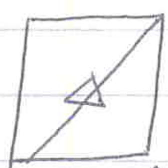
$ch : K(X) \hookrightarrow H^{ev}(X^{dual}; \mathbb{Q})$

- Max unipotent monodromy $\longleftrightarrow$ $\otimes \mathcal{L}_{gen}$ on K-theory

$$H^2(X^{dual}; \mathbb{C}) = \mathbb{C} = H^{1,1}(X)^{dual}$$

- dual to P-L (Picard-Lefschetz) of a vanishing cycle which is $S^3 \subset X$.

X^{dual}



X^{dual}

$(pr_1)_* \, pr_2^* \xi \otimes \mathcal{O}_\Delta$

$$ch(\xi) \longrightarrow ch(\xi) - \left(\int_{X^{dual}} ch(\xi) \cup Td(X^{dual}) \right) \cdot 1$$

$T_{PL} = 1 + N_{PL}$

$rk\ Im(N_{PL}) = 1$

$Im(N_{PL})$ is indivisible since $Td(X^{dual}) = 1 + \cdots$

Ω = nowhere zero section of $\mathcal{F}^3$

$\delta = z \dfrac{d}{dz}$

For the examples of complete intersections in toric varieties, get generalized hypergeom. eqns to describe Ω.

$$\delta^4\Omega + f_1(z)\delta^3\Omega + f_2(z)\delta^2\Omega + f_3(z)\delta\Omega + f_4(z)\Omega = 0.$$

$$f_i(z) = \frac{c_i z}{z-1}, \quad i = 1, \cdots, 4.$$

max unipotent monodromy at 0.

monod at ∞
$$\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ -f_4(\infty), & \cdots, & & -f_1(\infty) \end{pmatrix}$$

monod at 1 is unipotent rk 1.

<u>Question</u>: Inv. sympl form $\Leftrightarrow$ c_1 even integer

$\underbrace{\text{form det}}_{\text{up to const}}$ by c_1 & c_2

When is there an Inv. integral lattice unimodular (?)

<u>Question</u>: Is the max uni. mon. at 0 conj over $\mathbb{Z}$ to that predicted by MS?

Does ~~Is~~ the unipotent monod at 1 have indivisible nilp image over $\mathbb{Z}$?

Lemma. Rep of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$ with monod. at $0, 1$, conj (over $\mathbb{Z}$) to the MS predictions and q-unip at ∞ are determined up to conj by the char poly at ∞ & hence so is the variation of H.S.

Thus, if the hypergeometric differential equations which have invariant lattices have monodromy at 0 & 1 as predicted by MS (over $\mathbb{Z}$) then their V. of H.S. is determined by the monodromy at ∞. This would prove that these examples are as predicted by M.S. over $\mathbb{Z}$.