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Representations and characters for Lie superalgebras.

Simple Lie superalgebras (finite dim)

$\mathfrak{gl}(m|n) \supset \mathfrak{sl}(m|n)$ $\quad$ $\mathfrak{osp}(m|2n)$

$m \neq n$

$m = n$ $\quad \mathfrak{psl}(n|n)$

Exceptional

$D(2,1,\alpha)$, G_3, F_4

Contragradient.

Strange

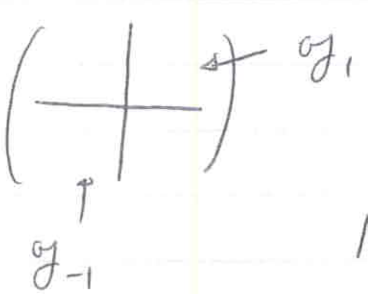
$Q(n)$

$P(n)$

Cartan type

$\mathfrak{g} = \mathfrak{gl}(m|n)$ $\qquad \mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$

$\mathbb{Z}$-grading compatible with $\mathbb{Z}_2$-grading



$\mathfrak{g} = \mathfrak{g}_{-1} \oplus (\mathfrak{g}_0 \oplus \mathfrak{g}_1)$.

M is a $\mathfrak{g}_0$-module $\rightsquigarrow$ $\mathfrak{g}_0 + \mathfrak{g}_1$-module

$\mathrm{Ind}_{\mathfrak{g}_0+\mathfrak{g}_1}^{\mathfrak{g}} M$

① Let M be an irred.

When is $\mathrm{Ind}_{\mathfrak{g}_0+\mathfrak{g}_1}^{\mathfrak{g}} M$ irreducible?

$\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M$.

$\mathfrak{g} \to \mathcal{U}(\mathfrak{g}) \supset Z$ - center. $\qquad \rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_0)} \alpha - \frac{1}{2}\sum_{\alpha \in \Delta(\mathfrak{g}_1)} \alpha$

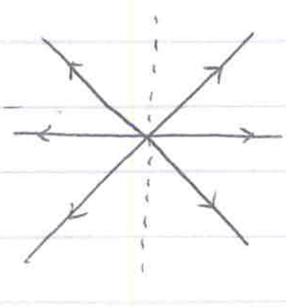
$h : Z \longrightarrow S(\mathfrak{h}) = \mathrm{Poly}(\mathfrak{h}^*)$ $\qquad \lambda \mapsto \lambda + \rho$ where ρ

- In classical case, $Z \simeq S(\mathfrak{h})^W$, $\quad W = S_m \times S_n$
- Z is not Noetherian. $\qquad \Delta(\mathfrak{g}_0) \quad \Delta(\mathfrak{g}_1) = -\Delta(\mathfrak{g}_1)$

<u>Theorem</u> (Kac, Sergeev)

$$Z = \{ f \in S(\mathfrak{h})^W \mid \lambda \in \mathfrak{h}^*, \text{ if } (\lambda, \alpha) = 0 \quad \alpha \in \Delta(\mathfrak{g}_1) \text{ then } f(\lambda + t\alpha) = f(\lambda) \ \forall t \}.$$

$\mathfrak{sl}(1|2)$



In $\mathfrak{g}$

$\downarrow$

Spec Z

$$\begin{tikzcd} \theta_\lambda : Z \arrow[r] \arrow[dr] & \mathbb{C} \\ & S(\mathfrak{h}) \arrow[u, "\text{eval at } \lambda"'] \end{tikzcd}$$

Def. λ <u>typical</u> if $(\lambda, \alpha) \neq 0$ for any $\alpha \in \Delta(\mathfrak{g}_1)$

- No complete reducibility

<u>Theorem</u> (Penkov, Gorelik) $\mathcal{U}_\lambda$-mod – category of $\mathfrak{g}$-mod with central character θ_λ

$\mathcal{U}_\lambda(\mathfrak{g}_0)$-mod — " — of $\mathfrak{g}_0$-mod — " —

functors:

$$\mathcal{U}_\lambda\text{-mod} \longrightarrow \mathcal{U}_\lambda(\mathfrak{g}_0)\text{-mod} \qquad M \mapsto M^{\mathfrak{g}_1}$$

$$\mathcal{U}_\lambda(\mathfrak{g}_0)\text{-mod} \longrightarrow \mathcal{U}_\lambda\text{-mod} \qquad N \mapsto \mathrm{Ind}_{\mathfrak{g}_0+\mathfrak{g}_1}^{\mathfrak{g}} N$$

If λ is typical, then these two functors give an equivalence of categories

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Let L_λ be an irred. $\mathfrak{g}$-module with h. weight λ

$L_\lambda(\mathfrak{g}_0)$ be — " — over $\mathfrak{g}_0$

$$K_\lambda = \mathrm{Ind}_{\mathfrak{g}_0+\mathfrak{g}_1}^{\mathfrak{g}} L_\lambda(\mathfrak{g}_0)$$

If λ is typical, then K_λ is irreducible.

As $\mathfrak{g}_0$-module, $K(\lambda) \simeq \mathcal{U}(\mathfrak{g}_{-1}) \otimes L_\lambda(\mathfrak{g}_0)$ for λ typical.

$$\mathrm{ch}\, L_\lambda = \mathcal{D} \sum_{w\in W} (-1)^w e^{w(\lambda)}$$

$$\mathcal{D} = \frac{\prod_{\alpha\in\Delta(\mathfrak{g}_1)} \left(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}\right)}{\prod_{\alpha\in\Delta^+(\mathfrak{g}_0)} \left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)}$$

Atypical case

$$(\lambda, \alpha_1) = (\lambda, \alpha_2) = \cdots = (\lambda, \alpha_k) = 0$$

λ is generic $\quad$ k is an invariant of Θ_λ and is called the degree of atypicality

$$S_\lambda = \{\alpha_1, \dots, \alpha_k\}.$$

Bernstein-Leites. $\quad$ λ generic

$$ch(L_\lambda) = \mathcal{D} \sum (-1)^w e^{w(\lambda)} w\left(\prod_{\alpha \in S_\lambda} (e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}})\right)$$

$ch\, L_\lambda$ for general λ.

$\mathcal{F}^{m|n}$ $\quad$ finite dimensional $\mathfrak{gl}(m|n)$-modules, semisimple over $\mathfrak{g}_0$ algebraic.

L_λ irred module $\qquad \lambda \in \Lambda^+(m|n)$

K_λ Kac module $\qquad \Updownarrow$

$$(a_m > a_{m-1} > \cdots > a_n \mid b_1 < \cdots < b_n,$$

$$a_i, b_j \in \mathbb{Z}$$

Each L_λ has a projective cover P_λ, which is a direct summand of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L_\lambda(\mathfrak{g}_0)$

$M^\vee := (M^*)^\sigma$ $\quad$ σ - Cartan involution.

$$P_\lambda^\vee \simeq P_\lambda, \quad L_\lambda^\vee \simeq L_\lambda$$

$[K_\lambda : L_\mu] = a_{\lambda\mu}$ Jintzen matrix

$$a_{\lambda\lambda} = 1$$

$$a_{\lambda\mu} \neq 0 \Rightarrow \mu \leq \lambda$$

- P_λ has a finite filtration with quotients isom. to Kac modules.

$(P_\lambda : K_\mu)$ is well defined

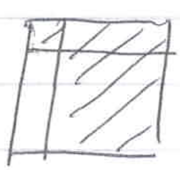
- $(P_\mu : K_\lambda) = [K_\lambda : L_\mu]$

$$\mathcal{F}^{1|0} \times \mathcal{F}^{m-1|n} \longrightarrow \mathcal{F}^{m|n} \qquad \text{Call } \Phi^m$$

$$\mathcal{F}^{m|n-1} \times \mathcal{F}^{0|1} \longrightarrow \mathcal{F}^{m|n} \qquad \Psi^n$$

derived functor

Geometric induction functor



$$\mathfrak{gl}(1) \oplus \mathfrak{gl}(m-1|n) \subset \mathfrak{p} \subset \mathfrak{gl}(m|n)$$

$$H(G/P, M) \leftarrow M$$

$$[\Phi_m(L_\lambda(\mathfrak{gl}(1) \oplus \mathfrak{gl}(m-1|n)))]$$

$$\Phi_m(\lambda, \mu) \in \mathbb{Z}[q]$$

Introduce coefficients

$$[H^i(G/P; L_\lambda) : L_\mu] = \Phi_m^i(\lambda, \mu)$$

$$\Phi_m\langle\lambda\rangle = \sum \Phi_m^i(\lambda, \mu)\, q^i$$

$\lambda = (a_m > a_{m-1} > \dots > a_1 \mid b_1 < b_2 < \dots)$

Relations on Φ_m

$a_m \neq b_j$

0. $\Phi_m \langle \lambda \rangle = 0$

$a_m = b_j$

$(a \; \dots \mid \dots \; a = b_j \; \dots)$

$\lambda' = (a-1 \; \dots \mid \dots \; a-1 \; \dots)$

1. $\lambda' \in \Lambda^+(m|n)$

$$\Phi_m \langle \lambda \rangle = \left[\Phi_m \langle \lambda' \rangle q^{-1} \right]_+ + q \langle \lambda' \rangle$$

2. $\Phi_m \langle \lambda \rangle = \Phi_{m-1} \langle \lambda' \rangle \quad (a \; a-1 \dots \mid \dots \; a \dots)$

3. $\Phi_m \langle \lambda \rangle = \Psi_j \langle \lambda' \rangle \quad (a \dots \mid \dots a-1 \; a \dots)$

4. $\Phi_m \langle \lambda \rangle = q \, \Phi_{m-1} \langle \lambda' \rangle \; (a \; a-1 \dots \mid \dots a-1 \; a \dots)$

<u>Theorem.</u> $\sum_\mu \Phi_m(\lambda, \mu) \langle \mu \rangle = \Phi_m \langle \lambda \rangle$

$$\operatorname{ch} L_\lambda = \sum_{\mu < \lambda} \prod_{i=1}^{m} (1 + \Phi_i)^{-1}_{q=-1} (\lambda, \mu) \operatorname{ch} K_\mu$$

J. Brundan (2002)

$(e_a, e_a) = 1$

$(e_a, e_b) = 0 \quad a \neq b$

$U_q(gl(\infty)) \qquad E_a, F_a, K_a, K_a^{-1}$

$Q(q) \qquad a \in \mathbb{Z}$

$K_a K_a^{-1} = 1 \qquad K_a K_b = K_b K_a$

$K_a E_b K_a^{-1} = q^{(e_a, e_b - e_{b+1})}$

$K_a F_b K_a^{-1} = q^{(e_a, e_{b+1} - e_b)}$

$E_a F_b - F_b E_a = \delta_{a,b} \left(\frac{K_a}{K_{a+1}} - \frac{K_{a+1}}{K_a} \right) / q - q^{-1}$

$E_a E_b = E_b E_a \quad \text{if } |a-b| > 1$

$E_a^2 E_b + E_b E_a^2 = (q + q^{-1}) E_a E_b E_a \quad \text{if } |a-b| = 1$

$\vdots$

$\Delta E_a = 1 \otimes E_a + E_a \otimes \frac{K_{a+1}}{K_a}$

$\Delta F_a = \frac{K_a}{K_{a+1}} \otimes F_a + F_a \otimes 1$

$\Delta K_a = K_a \otimes K_a$

$\mathcal{U}$- Modules $\quad \mathcal{V} \quad \{v_a\}_{a\in\mathbb{Z}}$

$$\mathcal{W} = \mathcal{V}^* \quad \{w_a\}_{a\in\mathbb{Z}}$$

$$K_a v_b = q^{\delta_{a,b}} v_b \qquad K_a w_b = q^{-\delta_{a,b}} w_b$$

$$E_a v_b = \delta_{a+1,b} v_a \qquad E_a w_b = \delta_{a,b} w_{a+1}$$

$$F_a v_b = \delta_{a,b} v_{a+1} \qquad F_a w_b = \delta_{a+1,b} w_a$$

$$\mathcal{T}^{m|n} := \mathcal{W}^{\otimes m} \otimes \mathcal{V}^{\otimes n} \xrightarrow[\text{completion}]{} \widehat{\mathcal{T}}^{m|n}$$

Basis of $\mathcal{T}^{m|n}$ could be numerated by $\lambda = (a_m \dots a_1 | b_1 \dots b_n)$

$$M_\lambda = w_{a_m} \otimes \dots \otimes w_{a_1} \otimes v_{b_1} \otimes \dots \otimes v_{b_n}$$

Example $\quad \lambda = (3, 2, 2 | 5, 7)$

$$M_\lambda = w_3 \otimes w_2 \otimes w_2 \otimes v_5 \otimes v_7$$

Hecke algebra $\mathcal{H}_{m|n}$

q-deformation of $S_m \times S_n$

$H_{-m+1}, \dots, H_{-1}, H_1, \dots, H_{n-1}$

$$H_i^2 = 1 - (q - q^{-1}) H_i$$

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}$$

$$H_i H_j = H_j H_i \quad \text{if} \quad |i-j| > 1$$

Right action of $\mathcal{H}_{m|n}$ on $\mathcal{T}^{m|n}$

$$M_\lambda H_i = \begin{cases} M_{\lambda \cdot s_i} & \text{if } \lambda < \lambda \cdot s_i \\ q^{-1} M_\lambda & \lambda = \lambda \cdot s_i \\ M_{\lambda \cdot s_i} - (q - q^{-1}) M_\lambda & \text{if } \lambda > \lambda \cdot s_i \end{cases}$$

The actions of $\mathcal{U}$ and $\mathcal{H}_{m|n}$ commute

$\mathcal{H}_{m|n}$ has a basis H_s, $s \in S_m \times S_n$

$$H = \sum_{s \in S_m \times S_n} (-q)^{\ell(s) - \ell(w_0)} H_s$$

a new $\mathcal{U}$-module

$$\mathcal{E}^{m|n} = \mathcal{T}^{m|n} \otimes_{\mathcal{H}} H$$

$$K_\lambda = M_\lambda \otimes_{\mathcal{H}} H \qquad \lambda \in \Lambda^+(m|n)$$

form a basis of $\mathcal{E}^{m|n} \rightsquigarrow \hat{\mathcal{E}}^{m|n}$

Lusztig $\quad \bar{\ }: q \to q^{-1}$

$$\mathcal{U} \to \mathcal{U}$$

$$\mathcal{H}^{m|n} \to \mathcal{H}^{m|n}$$

Possible to define the bar involution

$$\mathcal{T}^{m|n} \to \mathcal{T}^{m|n}$$

$$\overline{x \otimes h} = \bar{x} \otimes \bar{h} \qquad x \in \mathcal{U} \quad h \in \mathcal{H}^{m|n}$$

$$\overline{M}_\lambda = M_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(q,q^{-1}) M_\mu$$

Since $\overline{H} = H$ one can define

$$\bar{\ }: \mathcal{E}^{m|n} \to \mathcal{E}^{m|n}$$

and

$$\overline{K}_\lambda = K_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}(q,q^{-1}) K_\mu$$

Theorem. There exists unique topological bases $\{P_\lambda\}$ and $\{L_\lambda\}$ $(\lambda \in \Lambda^+(m|n))$ such that

- $\overline{P_\lambda} = P_\lambda$, $\overline{L_\lambda} = L_\lambda$
- $P_\lambda = K_\lambda + \sum_{\mu > \lambda} b_{\lambda,\mu} K_\mu$
- $L_\lambda = K_\lambda + \sum_{\mu < \lambda} a_{\lambda,\mu} K_\mu$

$q\mathbb{Z}[q] \ni a_{\lambda,\mu}$, $b_{\lambda,\mu} \in q^{-1}\mathbb{Z}[q^{-1}]$

$$ch\, L_\lambda = ch\, K_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu}(1)\, ch\, K_\mu$$

$$ch\, P_\lambda = ch\, K_\lambda + \sum_{\mu > \lambda} b_{-w_0\lambda,-w_0\mu}(1)\, ch\, K_\mu$$

Translation functors:

$M \mapsto (M \otimes E)_{\chi_a}$ $\qquad E_a$

$M \mapsto (M \otimes E^*)\chi_a$ $\qquad F_a$

Geometric induction:

$F_{110} \times \mathcal{F}_{m-1|n} \to \mathcal{F}_{m|n}$ $\qquad \mathcal{E}^{m-1|n} \to \mathcal{E}^{m|n}$

$v \mapsto (v_a \otimes v)H$