M. Gorelik

The center of P-type Lie superalgebra

$\mathfrak{g} = \mathfrak{sl}(m|n)$

$$\begin{array}{c|cc} & m & n \\ \hline m & & \mathfrak{g}_1 \\ n & \mathfrak{g}_{-1} & \end{array}$$

$\mathcal{U}(\mathfrak{g})$ u.e.

$Z(\mathfrak{g})$ center

$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$

$\mathfrak{n}^+ = \mathfrak{n}_0^+ + \mathfrak{g}_1$

$\mathfrak{n}^- = \mathfrak{n}_0^- + \mathfrak{g}_{-1}$

$M(\lambda)$ Verma module

$L(\lambda)$. simple quotient

$$M(\lambda) = \bigoplus_{d=0}^{\dim \mathfrak{g}_{-1}} M^{-d}(\lambda) \qquad \mathcal{U}(\mathfrak{g}) = \bigoplus_{k=-\dim \mathfrak{g}_{-1}}^{\dim \mathfrak{g}_1} \mathcal{U}(\mathfrak{g})_k$$

$M^0(\lambda) = M'(\lambda)$

$\mathfrak{g}_0$ Verma mod

$\mathfrak{g}_1, \mathfrak{g}_{-1}$ are comm

$\mathcal{U}(\mathfrak{g}_1) = \Lambda \mathfrak{g}_1$

① $v \in M^0(\lambda)$ is a $\mathfrak{g}_0$-primitive vector ($\mathfrak{n}_0^+ v = 0$)

$\Rightarrow$ v is a $\mathfrak{g}$-primitive vector

② y be a highest wt vector of $\mathfrak{g}_{-1}$
v_λ be a h. wt vector of $M(\lambda)$, $(\lambda, \beta) = 0$ $\beta =$ wt of y.
$y v_\lambda$ is a primitive vector.

③. $M(\lambda) = L(\lambda)$ if $\lambda \notin \bigcup H_k$

④ $\bigcap_{\lambda \in \mathfrak{h}^*} \operatorname{Ann} M(\lambda) = 0$

Take $0 \neq z \in Z(\mathfrak{g})$

$$h : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \Big/ \left(\mathcal{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- \mathcal{U}(\mathfrak{g})\right) = S(\mathfrak{h})$$

z acts on $M(\lambda)$ by $h(z)(\lambda)$ (from ③)

$h(z)$. nonzero (④)

$h(z) \in S(\mathfrak{h})^{W_0}$ (①)

if $(\lambda, \beta) = 0$ then ②.

$h(z)(\lambda) = h(z)(\lambda - t\beta)$

$\mathfrak{sl}(n|n)$

$$\begin{array}{ccc} & n & n \\ n & \left(\begin{array}{c|c} a & b \\ \hline c & -a^t \end{array}\right) & n \\ & \mathfrak{g}_1 & \mathfrak{g}_{-1} \end{array}$$

b is symm

c is skew-symm

$\operatorname{Tr} a = 0$.

$\mathfrak{n}^+ = \mathfrak{n}_0^+ + \mathfrak{g}_1$

$\mathfrak{n}^- = \mathfrak{n}_0^- + \mathfrak{g}_{-1}$

$\mathfrak{n}^+ = \mathfrak{n}_0^+ + \mathfrak{g}_{-1}$

"Big" Verma modules

$M(\lambda)$ is not irreducible

v_λ h. wt vector

$\underbrace{y_{2\varepsilon_1}}_{\in\, \mathfrak{g}_{-1}} v_\lambda$ is primitive

$\Delta_0^+ = \{ \varepsilon_i - \varepsilon_j \mid i < j \}$

$\Delta_1^+ = \Omega(\mathfrak{g}_1)$

$= \{ -\varepsilon_i - \varepsilon_j \}_{i \neq j}$

$\Delta_1^- = \Omega(\mathfrak{g}_{-1})$

$= \{ \varepsilon_i + \varepsilon_j, 2\varepsilon_i \}_{i \neq j}$

$$\Rightarrow \quad \left.\begin{array}{l} z \in Z(\mathfrak{g}) \\ h(z)(\lambda) = h(z)(\lambda - 2\varepsilon_1) \\ h(z) \text{ is } W\text{-inv.} \end{array}\right\} \Rightarrow h(z) \in \mathbb{C}.$$

Schevnert (1987) $\quad Z(\mathfrak{g}) = \mathbb{C} \oplus Z(\mathfrak{g})_{-n}$.

For $n = 3$, constructed elements.

($z^2 = 0$, if z homogeneous nonscalar).

the highest weight space of $M(\lambda)$ is not one-dimensional, since $\prod_{\beta \in \Delta_1^-} y_\beta$ has wt zero.

$a \in U(\mathfrak{g})$. is anticentral if a is odd and $au = ua \quad \forall u \in U(\mathfrak{g})$

or if a is even and $au = (-1)^{d(u)} ua, \quad u \in U(\mathfrak{g})$

$A(\mathfrak{g})$ is the set of anticentral elements

$$\phi : Z(\mathfrak{g}_0) \xrightarrow{\sim} A(\mathfrak{g})$$

$$c \longmapsto (\mathrm{ad}' X_J)(y_I c) \qquad y_I = \prod_{\beta \in \Delta_1^-} y_\beta \in \Lambda^{top} \mathfrak{g}_{-1}$$

$$X_J \in \Lambda^{top} \mathfrak{g}_1$$

Take a simple (graded) N.

denote by θ_N the linear map

$$\theta_N / N_{\bar 0} = id$$

$$\theta_N / N_{\bar 1} = -id$$

$a \in A(\mathfrak{g})$ acts.

on N by $\psi(a)$ on N. $\quad \psi(a) \in \mathrm{End}\, N$.

$\psi(a)\, \theta_N \in \mathrm{End}_{\mathfrak{g}} N$

For $M(\lambda)$

$\psi(a)$ $\qquad P: \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}_{-n} \longrightarrow S(\mathfrak{h})$

If $a \in A(\mathfrak{g})$. $\quad P(a) \neq 0$. $\quad (a \neq 0)$

$a_1, a_2 \neq 0 \qquad \dfrac{P(a_1)}{P(a_2)} \in \text{Fract } S(\mathfrak{h})^W$

$$\phi' : Z(\mathfrak{g}_0) \xrightarrow{\sim} Z(\mathfrak{g})$$

$$c \mapsto (\operatorname{ad} X_J)(y_I c)$$

$$P(\phi'(c)) = h(c) \cdot t$$

$$P : Z(\mathfrak{g}) \xrightarrow{\sim} t S(\mathfrak{h})^W$$

$$t = h(X_J y_J) = \prod_{\alpha \in \Delta_0^+} (\alpha^\vee + (\alpha, \rho_0) - 1)$$

Small Verma modules.

V. Serganova

$\bigcap \operatorname{Ann} M(\lambda) = I$ is the radical of $\mathcal{U}(\mathfrak{g})$.

$$\overline{\mathcal{U}} := \mathcal{U}(\mathfrak{g}) / I$$

$h : \overline{\mathcal{U}} \longrightarrow S(\mathfrak{h}) \qquad$ rest. to $Z(\overline{\mathcal{U}})$ inj.

$y_{-\varepsilon_{n-1} - \varepsilon_n} \qquad [y_{-\varepsilon_{n-1} - \varepsilon_n}, X_{\varepsilon_{n-1} + \varepsilon_n}] = \varepsilon_{n-1} - \varepsilon_n$.

$h(z)(\lambda) = h(z)(\lambda + t(\varepsilon_{n-1} + \varepsilon_n)) \qquad (\lambda, \varepsilon_{n-1} - \varepsilon_n) = 0$

Easy to show $\quad h(z) \in S \cdot S(\mathfrak{h})^W + \mathbb{C} \qquad S \in S(\mathfrak{h})$, $S = \prod_{\alpha \in \Delta_0} (\alpha^\vee + (\rho_0, \alpha) - 1)$

Theorem

$$h: Z(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C} + S \cdot S(\mathfrak{h})^W$$

Def. $\mu \in \mathfrak{h}^*$ is typical if $S(\mu) \neq 0$.

Cor. If μ is typical, and $\mathrm{Ann}_{Z(\mathfrak{g})} M(\mu) = \mathrm{Ann}_{Z(\mathfrak{g})} M(\lambda)$

then $\lambda \in W . \mu$

For typical μ we have equivalence of categories:

Take typical $\mu \in \mathfrak{h}^*$

χ - char of $M(\mu)$ $\quad (Z(\mathfrak{g}) \to \mathbb{C})$

χ' - char of $M'(\mu)$ $\quad (Z(\mathfrak{g}_0) \to \mathbb{C})$

$$\Psi : \underset{\text{has } \chi'}{N \cdot \mathfrak{g}_0\text{-mod}} \longrightarrow \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 + \mathfrak{g}_{-1}} N$$

$$\Phi : \underset{\chi}{M \ \mathfrak{g}\text{-mod},} \longrightarrow M^{\mathfrak{g}_{-1}}$$

$\Psi \ \Phi$ provide equivalence of categories

$\mathfrak{g}_0$-mod with character χ'

$\mathfrak{g}$-mod with character χ

$$\mathrm{ch}\, M = \mathrm{ch}\, M^{\mathfrak{g}_{-1}} \prod_{i \neq j} (1 - \varepsilon e^{\varepsilon_i + \varepsilon_j}) \qquad \varepsilon^2 = 1$$