# String Theory and del Pezzo Surfaces

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A. I

* A "duality" between string Theory (IIA/IIB) Compactified on a rectangular torus $T^R$ and objects in classical geometry

* The symmetries of the compactified string theory have geometric interpretation in classical geometry.

* This talk is incomplete since the microscopic understanding of the duality is not completely known.

* Objects in classical geometry that appear in this "duality" are 4-dim manifolds called del Pezzo Surfaces, $\mathbb{P}^2$ blown-up at $R$-points $R = 0, 1, 2, \ldots, 8$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

## Type IIA | IIB String Theory:

* Type IIA strings have target space $\mathbb{R}^{9,1}$. The space-time theory has 32 real supercharges. (i.e., maximally supersymmetric in 10-dim).

* Among the massless states of the theory are the anti-symmetric tensor fields.

$B^{(2)}$ NS-NS Sector (anti-periodic boundary conditions on the circle.)

$A^{(1)}, A^{(3)}, A^{(5)}$ RR Sector.

and their duals

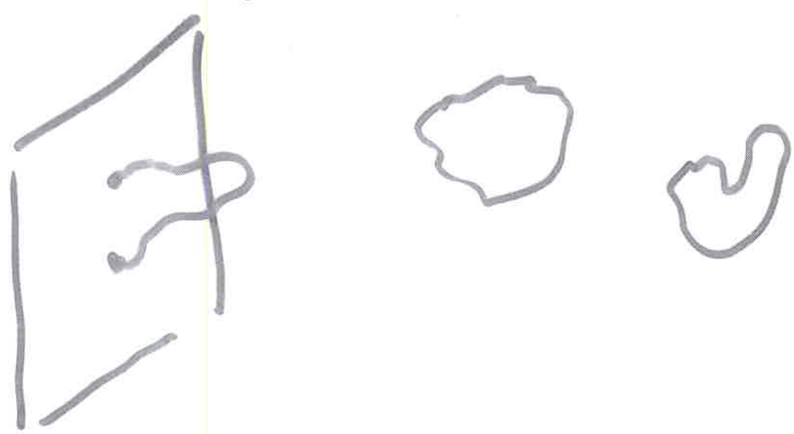
$(F = dA, \quad d\tilde{A} = *F)$

* The states which are charged under RR potentials are called D-branes

* The object charged under the B-field $B^{(2)}$ is the type IIA string itself.

* The object charged under the dual of the B-field is a 5+1 dim object called the NS-5brane.

* A Dp-brane is a $(p+1)$ dim subspace of $\mathbb{R}^{9,1}$ on which open strings can end



Thus in type IIA we have the following set of D-branes.

| | | |
|---|---|---|
| D0-brane | 0+1 | dimensional |
| F-string | 1+1 | " |
| D2-brane | 2+1 | " |
| D4-brane | 4+1 | " |
| NS 5-brane | 5+1 | " |
| D6-brane | 6+1 | " |
| D8-brane | 8+1 | " |

Since presence of these objects breaks translational invariance the supersymmetry is also broken. These are $\frac{1}{2}$BPS objects i.e., they preserve 16 supercharges.

# Type IIB string theory:

Type IIB strings also propagate in 10-dim $\mathbb{R}^{9,1}$. The space-time theory has 32 real supercharges but is also chiral, (2,0).

The difference with IIA strings lies in the massless states of the theory.

NS-NS $\quad B^{(2)}$

RR-sector $\quad \chi^{(0)}, A^{(2)}, A^{(4)}_{*}$

and their duals.

## Type IIB branes:

| | |
|---|---|
| F-string | 1+1 dim |
| D-string | 1+1 dim |
| D3-brane | 3+1 dim |
| D5-brane | 5+1 dim |
| NS5-brane | 5+1 dim |
| D7-brane | 7+1 dim |
| NS7-brane | 7+1 dim. |

Type IIB string theory has $SL(2,\mathbb{Z})$ symmetry. Under the $S \in SL(2,\mathbb{Z})$ ($S^2 \equiv -1$) the D-branes & the NS-branes are exchanged. The D3-brane, however, is invariant.

* Type IIA and IIB string theories are different in 10-dimensions. However, in nine-dimensions & lower they become equivalent.

$$\text{Type IIA on } \mathbb{R}^{9-k,1} \times T^k$$
$$\cong \text{Type IIB on } \mathbb{R}^{9-k,1} \times \tilde{T}^k$$

$k \geq 1.$

T-duality.

$$\text{Type IIA on } \mathbb{R}^{8,1} \times S^1 \cong \text{Type IIB on } \mathbb{R}^{8,1} \times S^{1\prime}$$

Radius of $S^1 = R$

Radius of $S^{1\prime} = 1/R$

M-Theory: M-theory is the mysterious 11-dimensional theory whose low energy limit is the $D=11$ Supergravity. Also M-theory is the strong coupling limit of Type IIA strings.

All string theories in 10-dimensions can be obtained from M-theory in various limits

* Type IIA strings in $D=10$

$\cong$ M-theory on $S^1 \times \mathbb{R}^{9,1}$

$$R = g_s l_s \,, \quad l_p = 2\pi g_s^{\frac{1}{3}} l_s .$$

$$R = g_s^{2/3} \, l_P / 2\pi$$

11-dim Supergravity has an anti-symmetric 3-form field $C^{(3)}$.

The object charged under this is (2+1) dim called M2-brane

The dual of the 3-form field is a six-form field $\tilde{C}^{(6)}$ and the object charged under this is the M5-brane, a 5+1 dim object.

The tension (Mass per unit volume) is given by

$$T_{M2} = 2\pi / l_p^3$$

$$T_{M5} = 2\pi / l_p^6$$

| | | |
|---|---|---|
| $D=11$ | M-Theory | |
| $D=10$ | Type IIA ; Type IIB ($\mathbb{Z}_2$) | |
| $D=9$ | IIA/IIB | $\mathbb{Z}_2$ |
| $D=8$ | IIA/IIB | $S_3 \times \mathbb{Z}_2$ |
| $D=7$ | IIA/IIB | $S_4$ |
| $D=6$ | IIA/IIB | $W(E_5)$ |
| $D=5$ | IIA/IIB | $W(E_6)$ |
| $D=4$ | IIA/IIB | $W(E_7)$ |
| $D=3$ | IIA/IIB | $W(E_8)$ |

U-duality group in $11-k$ dim
is $W(E_k)$

# Del Pezzo Surfaces:

Compact complex manifolds of positive first Chern class.

- $\mathbb{P}^2$
- $\mathbb{P}^2$ blown-up at one point ; $\mathbb{P}^1 \times \mathbb{P}^1$
- $\mathbb{P}^2$ blown up at $k$-points, $k = 2, \ldots, 8$.

$$\mathbb{P}^2 \text{ blown-up at } k\text{-points} = \mathbb{B}_k$$

$$\mathbb{B}_k \cong (\mathbb{P}^1 \times \mathbb{P}^1)_{k-1}, \quad k = 2, \ldots, 8.$$

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$$\begin{array}{lll}
\mathbb{P}^2 & & K^2 = 9 \\
\downarrow & & \\
\mathbb{B}_1 & \mathbb{P}^1 \times \mathbb{P}^1 & K^2 = 8 \\
\downarrow & \swarrow & \\
\mathbb{B}_2 & & K^2 = 7 \\
\downarrow & & \\
\mathbb{B}_3 & & K^2 = 6 \\
\downarrow & & \\
\mathbb{B}_4 & & K^2 = 5 \\
\vdots & & \\
\mathbb{B}_8 & & K^2 = 1
\end{array}$$

$\underline{H_2(\mathbb{B}_k, \mathbb{Z})}$: A basis is given by

$$\{H, E_1, \dots, E_k\}$$

$$H^2 = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad H \cdot E_i = 0$$

$$i, j = 1, \dots, k.$$

$$K_{\mathbb{B}_k} = -3H + \sum_{i=1}^{k} E_i$$

$$K^2_{\mathbb{B}_k} = 9 - k$$

If $C \in H_2(\mathbb{B}_k, \mathbb{Z})$ then degree of $C$ is defined as

$$d_C = -K_{\mathbb{B}_k} \cdot C$$

$$C^2 = 2g - 2 + d_C$$

$g$ = genus of the curve realizing the class $C$.

$H_2(B_k, \mathbb{Z})$ contains a codimension one lattice isomorphic to the root lattice of $E_k$. The set of roots is defined as follows.

$$R = \{ \alpha \in H_2(B_k, \mathbb{Z}) \mid \alpha^2 = -2,\ d_\alpha = 0 \}$$

a set of simple roots is given by

$$\alpha_i = E_i - E_{i+1}, \quad i = 1, \ldots, k-1$$

$$\alpha_k = H - E_1 - E_2 - E_3. \qquad (k \geq 3).$$

$$\alpha_a \cdot \alpha_b = -A_{ab} \qquad a, b = 1, \ldots, k$$

$A$ = Cartan matrix of the Lie algebra $E_k$

$k=2$  $A_1$

$k=3$  $A_2 \times A_1$

$k=4$  $A_4$

$k=5$  $D_5$

$k=6$  $E_6$

$k=7$  $E_7$

$k=8$  $E_8$

Dynkin diagrams of $E_k$

Given this set of simple roots to each $C \in H_2(\mathbb{B}_k, \mathbb{Z})$ we can associate an $E_k$ weight vector $\lambda$

$$\lambda_a = - C \cdot \alpha_a \qquad a = 1, \dots, k$$

$$C^2 = -\lambda^2 + \frac{d_C^2}{9-k}$$

The lattice $H_2(\mathbb{B}_k, \mathbb{Z})$ has signature $(1,k)$

$$H_2(\mathbb{B}_k, \mathbb{Z}) = (K_{\mathbb{B}_k}) \oplus \Gamma_{E_k}$$

root lattice of $E_k$

Weyl group action:

Given any $\alpha \in H_2(\mathbb{B}_k, \mathbb{Z})$ with $\alpha^2 = -2$, $K_{\mathbb{B}_k} \cdot \alpha = 0$ we can define a transformation $w_\alpha$

$$w_\alpha : C \longrightarrow C + (C \cdot \alpha)\alpha = C'$$

$$C \in H_2(\mathbb{B}_k)$$

$$C' \cdot C' = C \cdot C$$

$$d_{C'} = d_C$$

$$\lambda_C \rightarrow w_\alpha(\lambda_C)$$

* for $\alpha_i = E_i - E_{i+1}$ this corresponds to exchanging the two blown-up points.
* Since $d_C$ is invariant, Curves of a given degree form a representation of the Weyl group.

Rational Curves:

$$g=0$$

$$\Rightarrow \quad C^2 = d_c - 2$$

$$C = nH - \sum_{a=1}^{k} m_a E_a$$

$$n(n-3) - \sum_{a=1}^{k} m_a(m_a - 1) = -2$$

<u>Example</u>: $d_c = 1$ classes in $H_2(B_8, \mathbb{Z})$

$$E_a, \quad H - E_a - E_b, \quad 2H - \sum_{i=1}^{5} E_{a_i}$$

$$3H - 2E_a - \sum_{i=1}^{6} E_{a_i}$$

$$4H - 2E_a - 2E_b - 2E_c - \sum_{i=1}^{5} E_{a_i}$$

$$5H - 2\sum_{i=1}^{6} E_{a_i} - \sum_{j=1}^{2} E_{a_j}$$

$$6H - 3E_a - 2\sum_{i=1}^{7} E_{a_i}$$

In terms of the $E_k$ weight vector

$$\lambda^2 = +\frac{d_c^2}{9-k} - d_c + 2$$

for $d_c = 1$

$$\lambda^2 = \frac{10-k}{9-k}$$

This $d_c=1$ rational Curves are in the fundamental rep of $E_k$.

$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at $(k-1)$ points:

$$\{ l_1, l_2, e_1, \dots, e_{k-1} \}$$

$$l_1^2 = 0 \,, \quad l_2^2 = 0 \,, \quad l_1 \cdot l_2 = +1$$

$$e_i \cdot e_j = -\delta_{ij} \,, \quad l_1 \cdot e_i = l_2 \cdot e_i = 0$$

$$i, j = 1, \dots, k-1 .$$

$$c_1 = -K = 2l_1 + 2l_2 - \sum_{a=1}^{k-1} e_a$$

$$H \longmapsto l_1 + l_2 - e_1$$

$$E_1 \longmapsto l_2 - e_1$$

$$E_2 \longmapsto l_1 - e_1$$

$$E_{a+1} \longmapsto e_a \,, \quad a = 2, \dots, k-1$$

- - - - - - - - - - .

$$l_1 \longrightarrow H - E_1$$

$$l_2 \longrightarrow H - E_2$$

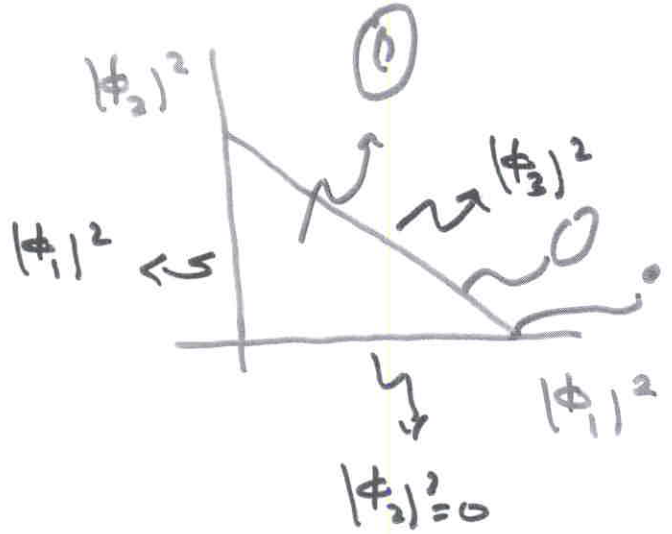
$$e_1 \longrightarrow H - E_1 - E_2$$

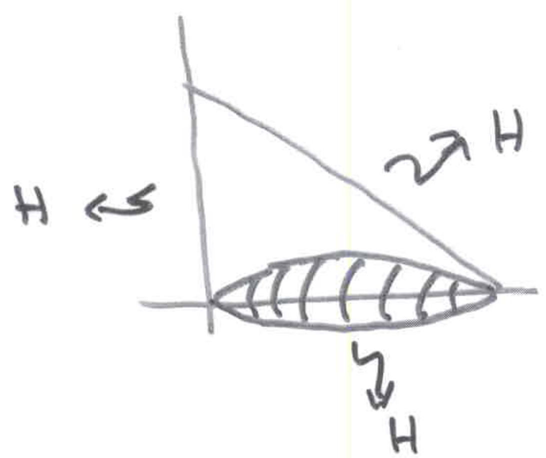
$$e_a \longrightarrow E_{a+1} \,, \quad a = 2, \dots, k-1$$

Toric geometry:

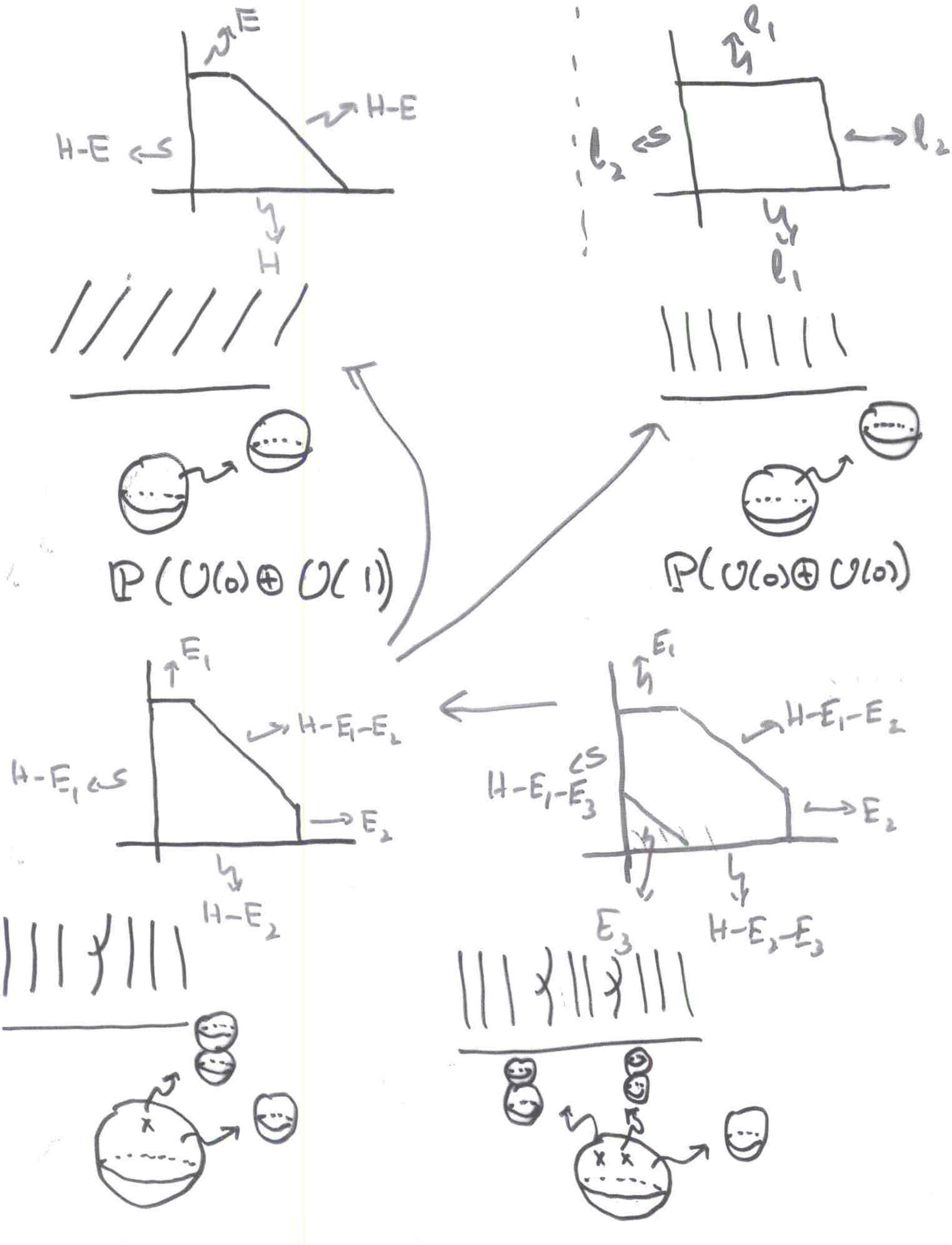
$\mathbb{P}^2$: $\{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = 1\} / U(1)$.

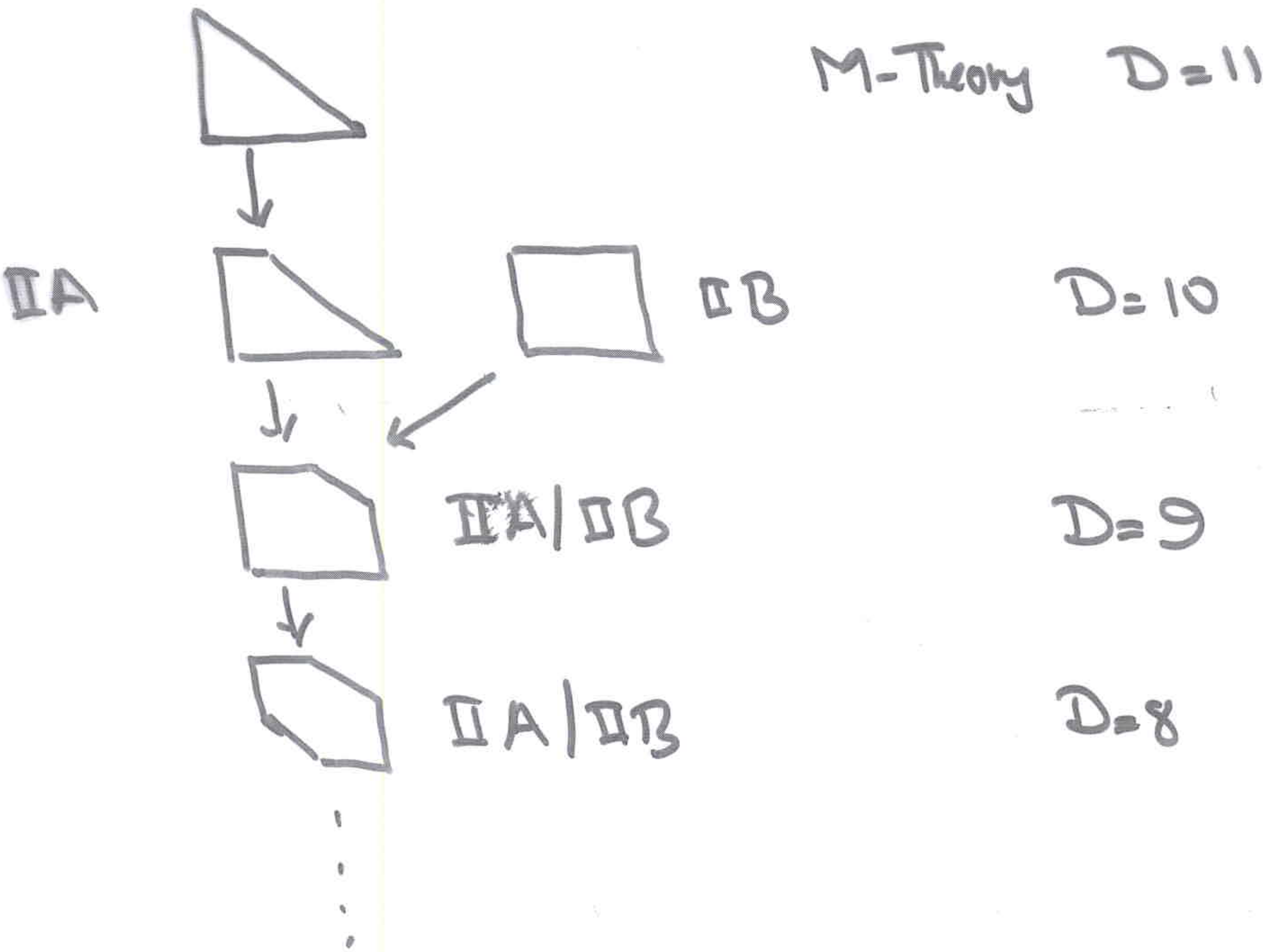
$$(\phi_1, \phi_2, \phi_3) \simeq (\phi_1 e^{i\theta}, \phi_2 e^{i\theta}, \phi_3 e^{i\theta})$$



$\mathbb{P}^2$ blown-up at one point $\mathcal{B}_1$ | $\mathbb{P}^1 \times \mathbb{P}^1$




M-Theory D=11
IIA
IIB
D=10
IIA/IIB
D=9
IIA/IIB
D=8

# Summary

| del Pezzo $B_k$ | M-Theory on $T^k$ |
|---|---|
| Elements of $H^2(B_k, \mathbb{R})$ | Points in moduli space of M-Theory on $T^k$. |
| Global diffeomorphisms preserving the canonical class $K$ | U-duality group |
| 2-sphere $C$ with volume $V_c$ and degree $p+1$ | $\frac{1}{2}$BPS $p$-brane state with Tension $2\pi \exp V_c$ |
| Volume of canonical class $V_K$ | Compactified Planck length |
| Volume $V_H$ of hyperplane class $H$. | 11-dim Planck length $l_p^{-3} = \exp V_H$ |
| Volume $V_E$ of exceptional curve $E$ | Radius $2\pi R = \exp -V_E$ |
| $H$, line in $\mathbb{P}^2$ | M2-brane |
| $2H$, Conic in $\mathbb{P}^2$ | M5-brane |
| 2-spheres $C_1, C_2$ $C_1 + C_2 = -K$ | Electric-Magnetic dual objects |

U-duality Group: The U-duality group of M-Theory on $T^k$ (rectangular with no C-field) is the Weyl group of $E_k$.

* If $R_i$ are the radii of the $T^k$ then U-duality group is generated by the following transformations

$$\omega_{\alpha_i} : R_i \longleftrightarrow R_{i+1} \quad i=1,\dots,k-1$$

and for $k \geq 3$.

$$\omega_{\alpha_k} : 2\pi R_1 \longrightarrow \frac{l_p^3}{2\pi R_2 \, 2\pi R_3}$$

$$2\pi R_2 \longrightarrow l_p^3 / 2\pi R_1 \, 2\pi R_3$$

$$2\pi R_3 \longrightarrow l_p^3 / 2\pi R_1 \, 2\pi R_2$$

$$l_p^3 \longrightarrow l_p^6 / 2\pi R_1 \, 2\pi R_2 \, 2\pi R_3$$

Naive Moduli Space $\widehat{\mathcal{M}}_k = \mathbb{R}_+^{k+1}$

$$\mathcal{M}_k = \widehat{\mathcal{M}}_k / W(E_k)$$

$W(E_k)$ acts linearly on

$$(\log \ell_p, \log R_1 2\pi, \log 2\pi R_2, \dots, \log 2\pi R_k)$$

$$\omega \in H^2(\mathbb{B}_k, \mathbb{R})$$

$$\omega(H) = +3 \log \ell_p$$

$$\omega(E_a) = +\log(2\pi R_a) \;;\; a = 1, \dots, k.$$

$\omega$ = "Kähler Form".

## (M-Theory in D=11 ; $\mathbb{P}^2$)

$$C = nH$$

$$g(C) = \frac{(n-1)(n-2)}{2}$$

There are two rational Curves

$H, \quad 2H$

$d_H = 3 =$ world-volume dimension of M2-brane

$d_{2H} = 6 =$ world-volume dimension of M5-brane

There are no other $\frac{1}{2}$BPS M-branes in M-theory.

$$T(\text{M2-brane}) = \frac{2\pi}{\ell_p^3} = 2\pi \exp(-V_H) = 2\pi \exp(-\omega(H))$$

$$T(\text{M5-brane}) = \frac{2\pi}{\ell_p^6} = 2\pi \exp(-V_{2H}) = 2\pi \exp(-\omega(2H))$$

(Type IIA in $D=10$ ; $\mathbb{B}_1$)

$$C = nH - mE$$

$$n(n-3) - m(m-1) = -2$$

$$(n,m) = \left(\frac{p}{2}, \frac{p}{2} - 1\right) \qquad p \in 2\mathbb{Z}$$

$$= \left(\frac{p+3}{4}, -\frac{p-5}{4}\right) \qquad p \in 4\mathbb{Z}+1$$

$$0 \leq p \leq 8$$

| | | | |
|---|---|---|---|
| $E$ | $d_E = 1$ | D0-brane | $T = 1/R$ |
| $H-E$ | $d = 2$ | F-string | $T = (2\pi)^2 R / \ell_p^3$ |
| $H$ | $d = 3$ | D2-brane | $T = 2\pi / \ell_p^3$ |
| $2H-E$ | $d = 5$ | D4-brane | $T = (2\pi)^2 R / \ell_p^6$ |
| $2H$ | $d = 6$ | NS5-brane | $T = 2\pi / \ell_p^6$ |
| $3H-2E$ | $d = 7$ | D6-brane | $T = (2\pi)^3 R^2 / \ell_p^9$ |
| $4H-3E$ | $d = 9$ | D8-brane | $T = (2\pi)^4 R^3 / \ell_p^{12}$ |

$$T_{Dp\text{-brane}} = T_{\frac{p}{2},\frac{p}{2}-1} = \frac{(2\pi)^{p/2} R^{\frac{p}{2}-1}}{l_p^{3p/2}}$$

$$T_{NS\ p\text{-brane}} = T_{\frac{p+3}{4},\frac{5-p}{4}} = \frac{(2\pi)^{\frac{9-p}{4}} R^{\frac{5-p}{4}}}{l_p^{\frac{2p+9}{4}}}$$

$$C_{\text{Electric}} + C_{\text{MAGNETIC}} = -K = 3H - E$$

<u>example</u>:

$$\underbrace{H-E}_{\text{string}} + \underbrace{2H}_{\text{NS 5-brane}} = 3H - E$$

$$\underbrace{H}_{\text{D2-brane}} + \underbrace{2H-E}_{\text{D4-brane}} = 3H - E$$

(Type IIB in $D=10$ ; $\mathbb{P}^1 \times \mathbb{P}^1$)

$$C = n\ell_1 + m\ell_2$$
$$d_C = p+1$$

$$2nm = p-1 \quad , \quad 2(n+m) = p+1$$

$$(n,m) = \left(1, \frac{p-1}{2}\right) \qquad p \in 2\mathbb{Z}+1$$
$$= \left(\frac{p-1}{2}, 1\right)$$

$$T_{Dp\text{-brane}} = \frac{1}{(2\pi)^{\frac{p-1}{2}}} T_{F\text{-string}}^{\frac{p-1}{2}} T_{D\text{-string}}$$

$$T_{NS\ p\text{-brane}} = \frac{1}{(2\pi)^{\frac{p-1}{2}}} T_{F\text{-string}} T_{D\text{-string}}^{\frac{p-1}{2}}$$

$0 \leq p \leq 8$

| | | | |
|---|---|---|---|
| $d=2$ | $\ell_1$ | F-string | $T = 1/(2\pi \ell_s)^2$ |
| $d=2$ | $\ell_2$ | D-string | $T = \dfrac{1}{(2\pi \ell_s)^2 g_s}$ |
| $d=4$ | $\ell_1+\ell_2$ | D3-brane | $T = \dfrac{1}{(2\pi \ell_s)^4 g_s}$ |
| $d=6$ | $2\ell_1+\ell_2$ | D5-brane | $T = \dfrac{1}{(2\pi \ell_s)^6 g_s}$ |
| $d=6$ | $\ell_1+2\ell_2$ | NS 5-brane | $T = \dfrac{1}{(2\pi \ell_s)^6 g_s^2}$ |
| $d=8$ | $3\ell_1+\ell_2$ | D7-brane | $T = \dfrac{1}{(2\pi \ell_s)^8 g_s}$ |
| $d=8$ | $\ell_1+3\ell_2$ | NS 7-brane | $T = \dfrac{1}{(2\pi \ell_s)^8 g_s^3}$ |

$$-K = 2\ell_1 + 2\ell_2$$

$\Rightarrow$ $d_c$ is always even.

* B. Julia et al (hep-th/0203 ) recently showed that space-time equations of motion for the anti-symmetric tensor fields can be obtained from a generalized Kac-Moody algebra based on $H_2(B_k, \mathbb{Z})$.

* Also anti-symmetric tensor fields living on the D-branes (or NS-branes) can also be deduced from the del pezzo picture and their world volume equations of motions can be ~~obj~~ obtained.

* Work in progress to understand the physical picture behind this correspondence.