# $N = 2$ Supersymmetry of Super KP Equations

1. Supercurves $M$, their duals $\hat{M}$, and untwisted $N = 2$ super Riemann surfaces $M_2$ (the geometry of hidden $N = 2$ superconformal symmetry in 2d).

2. $N = 1$ and $N = 2$ super KP hierarchies.

3. Functions and line bundles on $M, \hat{M}, M_2$.

4. In progress.


F. Delduc and L. Gallot, *Commun. Math. Phys.* **190** (1997) 395 – 410.

M. Bergvelt and JMR, *Duke Math. J.* **98** (1999) 1–57.

F. Ongay and JMR, hep-th/0203174 and in progress.

## 1. $M$, $\hat{M}$, and $M_2$

$M$ smooth supercurve or $1|1$ complex supermanifold: Riemann surface $M_{\mathrm{red}}$ with compatible transition functions

$$z' = F(z,\theta), \quad \theta' = \Psi(z,\theta)$$

in $\Lambda[z,\theta]$, where $\Lambda =$ complex Grassmann algebra on "constant" generators $\beta_1, \beta_2, \ldots, \beta_q$.

Set $\beta_i = 0$ to get $M_{\mathrm{sp}}$.
$z' = F(z)$, $\theta' = \theta\psi(z)$.
Functions on $M_{\mathrm{sp}}$ are functions $f(z)$ on $M_{\mathrm{red}}$ and $\theta g(z)$ for sections $g(z)$ of line bundle $\mathcal{N}$.

Obtain $M_2$ by adding coordinate $\rho$:

$$\rho' = \mathrm{ber}\begin{bmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{bmatrix} \rho + \frac{\partial_\theta F}{\partial_\theta \Psi},$$

[Dolgikh, Rosly, Schwarz 1990]

SRS since $D^+ = \partial_\rho$, $D^- = \partial_\theta + \rho\partial_z$ generate odd subbundle of $T(M_2)$ and $\{D^+, D^-\} = 2\partial_z$. Untwisted since $D^\pm$ each generate line subbundle.

$$0 \to \mathcal{O} \xrightarrow{\text{inc}} \mathcal{O}_2 \xrightarrow{D^+} \mathrm{Ber} \to 0.$$

Set $u = z - \theta\rho$ to get dual curve $\hat{M}$:

$$u' = \hat{F}(u, \rho) = \left[F + \frac{DF}{D\Psi}\right](u, \rho),$$

$$\rho' = \hat{\Psi}(u, \rho) = \frac{DF}{D\Psi}(u, \rho),$$

$$D = \partial_\theta + \theta\partial_z.$$

$$0 \to \hat{\mathcal{O}} \xrightarrow{\text{inc}} \mathcal{O}_2 \xrightarrow{D^-} \hat{\mathrm{Ber}} \to 0.$$

$D^\pm$ are superconformal derivatives and $M$, $\hat{M}$ are Spec of chiral and antichiral functions on $M_2$.

$D^-$ acts as $D$ on $\mathcal{O} \subset \mathcal{O}_2$.

$\Lambda$-points $(z_0, \theta_0)$ of $\hat{M}$ are parameters of irreducible divisors $z - z_0 - \theta\theta_0$ on $M$.

$M$ is an $N = 1$ SRS iff $M = \hat{M}$.

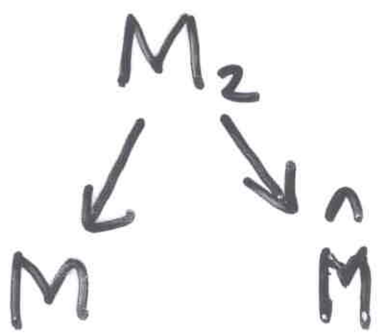
$\hat{M}_{\text{sp}}$ is $(M_{\text{red}}, K\mathcal{N}^{-1})$, generalizing Serre duality.

$M_2$ is common projectivized cotangent bundle of $M$, $\hat{M}$.

$M_2$ has trivial Ber bundle.

Sections of $\hat{\text{Ber}}$ are invariantly integrated over paths with endpoints $\Lambda$-points of $M$, locally $F(2) - F(1)$ with $\hat{\omega} = DF$; integration is odd.

In split case, $\hat{\text{Ber}} = \mathcal{N}|K$.



## 2. Jacobian super KP Hierarchy

JMR 1991, Mulase 1991

Describes linear flow of initial line bundle $\mathcal{L}_0$ through Jacobian of $M$.

Cover $M$ by disk $U$ around $P(z = 0)$ and $M - P$, overlapping in annulus $A$.

Line bundle $\mathcal{L}(t, \tau)$ has transition function

$$\exp(t_i z^{-i} + \tau_i \theta z^{-i+1})$$

on $A$. Flow is $\mathcal{L}_0 \mapsto \mathcal{L}_0 \otimes \mathcal{L}(t, \tau)$.

Lax pair description requires *Baker function* $w$: unique section of $\mathcal{L}$ holomorphic except for $z^{-1}$ pole at $P$. This requires

$$H^0(M, \mathcal{L}) = H^1(M, \mathcal{L}) = 0,$$

implying $\deg \mathcal{L} = g - 1$ and $\deg \mathcal{N} = 0$; $M$ is an *SKP curve*, *generic* if also $\mathcal{N} \neq \mathcal{O}$.

Functions $f$ on $M - P$ correspond to differential operators $Q(x, \zeta)$ by $fw = Qw$; $Q$ acts on $\exp(xz^{-1} + \zeta\theta + t_i z^{-i} + \tau_i \theta z^{-i+1})$ in $w$ written in chart $M - P$.

If $z^{-n}$ extends to an otherwise holomorphic function on $M$, corresponding operator $L = S\partial_x^n S^{-1}$ evolves by

$$\frac{\partial L}{\partial t_k} = [L_+^{k/n}, L],$$

$$\frac{\partial S}{\partial t_k} = -(S\partial_x^k S^{-1})_- S,$$

$$\frac{\partial S}{\partial \tau_k} = -(S\partial_\zeta \partial_x^{k-1} S^{-1})_- S.$$

$$S = 1 + a_1 D^{-1} + a_2 D^{-2} + \cdots$$

# N=2 Super KP

Delduc & Gallot 1997

Correspondence between even $N = 2$ differential operators

$$\breve{L} = D_x^+ \sum_{i=0}^{n} u_i(x, \zeta, \eta) \partial_x^i \; D_x^-$$

and $N = 1$ operators without constant term

$$\underline{L} = \sum_{i=0}^{n} \{[u_i^{(0)} + \zeta u_i^{(1)}] \partial_x^{i+1} + u_i^{(1)} D_x \partial_x^i\},$$

where $u_i(x, \zeta, \eta) = u_i^{(0)}(u, \zeta) + \eta u_i^{(1)}(x, \zeta)$.

Flows

$$\frac{\partial \breve{L}}{\partial t_k} = [\breve{L}^{k/n}_{+}, \breve{L}]$$

correspond to nonstandard SKP,

$$\frac{\partial \underline{L}}{\partial t_k} = [\underline{L}^{k/n}_{\geq 0}, \underline{L}],$$

equivalent to standard SKP via

$$\underline{L} = e^{\phi} L e^{-\phi},$$

a change of trivialization on $M$.

Correspondence is simply lifting from $M = \mathbf{C}^{1|1}_{(x,\zeta)}$ to $M_2 = \mathbf{C}^{1|2}_{(x,\zeta,\eta)}$; $\breve{L}$ agrees with $\underline{L}$ on chiral functions and annihilates antichiral functions.

## 3. Functions and Line Bundles on $M, \hat{M}, M_2$

Generic SKP curves have free cohomology:

$$\begin{array}{ll} H^0(M, \mathcal{O}) = \Lambda|0 & H^1(M, \mathrm{Ber}) = 0|\Lambda \\ H^1(M, \mathcal{O}) = \Lambda^g|\Lambda^{g-1} & H^0(M, \mathrm{Ber}) = \Lambda^{g-1}|\Lambda^g \end{array}$$

Split curves *always* have free cohomology, e.g.

$$H^0(M_{\mathrm{sp}}, \mathcal{O}) = H^0(M_{\mathrm{sp}}, \mathcal{O}|\mathcal{N}) = \Lambda|0.$$

$H^0(M, \bullet)$ is generally a submodule of $H^0(M_{\mathrm{sp}}, \bullet)$; $H^1(M, \bullet)$ is generally a quotient.

Cohomologies of $\hat{M}$, $M_2$ are generally not free:

$$\begin{array}{ll} H^0(\hat{M}, \hat{\mathcal{O}}) \leq \Lambda|\Lambda^{g-1} & H^1(\hat{M}, \hat{\mathrm{Ber}}) \leq \Lambda^{g-1}|\Lambda \\ H^1(\hat{M}, \hat{\mathcal{O}}) \leq \Lambda^g|0 & H^0(\hat{M}, \hat{\mathrm{Ber}}) \leq 0|\Lambda^g \end{array}$$

$$H^0(M_2, \mathcal{O}_2) \leq \Lambda^{g+1}|\Lambda^{g-1}$$

$$H^1(M_2, \mathcal{O}_2) \leq \Lambda^{g+1}|\Lambda^{g-1}$$

Functions on $\hat{M}$ come from integrating sections of Ber on $M$; odd period matrix controls which are single-valued.

Functions on $M_2$ come from lifting the $\Lambda|0$ and $\Lambda|\Lambda^{g-1}$ functions on $M, \hat{M}$, but also from lifting pairs of multivalued functions having opposite periods.

Some bundles on $M_2$ come from $\otimes$ lifts of pairs of bundles $\Lambda^g|\Lambda^{g-1}$ and $\Lambda^g|0$ from $M, \hat{M}$. (Degrees add.) Nothing new from $\hat{M}$ – maybe less! – so NOT all of Jac($M_2$) is obtained this way. Extra bundle has transition function in $A$

$$\exp -k\theta\rho/z = 1 - k\frac{\theta\rho}{z} = z^{-k}(z-\theta\rho)^k$$

and is not generated by SKP flows from $M$.

In general (don't assume $M$ is SKP) there is an obstruction to factoring a bundle on $M_2$ as $\otimes$ lifts of bundles from $M, \hat{M}$.

$$c \longmapsto c \times c^{-1}$$

$$0 \to \Lambda^\times_{\text{ev}} \longrightarrow \mathcal{O}^\times_{\text{ev}} \times \hat{\mathcal{O}}^\times_{\text{ev}} \longrightarrow \mathcal{O}^\times_{2,\text{ev}} \to 0$$

$$F \times \hat{F} \longmapsto F\hat{F}$$

$$H^1(\mathcal{O}^\times_{\text{ev}} \times \hat{\mathcal{O}}^\times_{\text{ev}}) \xrightarrow{\alpha} H^1(\mathcal{O}^\times_{2,\text{ev}}) \xrightarrow{\beta} H^2(\Lambda^\times_{\text{ev}}).$$

Im $\alpha$ is bundles that factor; same as Ker $\beta$.

More concretely, bundle transition functions $\Gamma_{ij}$ on $M_2$ always factor locally as $F_{ij}\hat{F}_{ij}$ up to multiplicative constants $c_{ij}, c_{ij}^{-1}$. Cocycle condition

$$(F_{ij}F_{jk}F_{ki})(\hat{F}_{ij}\hat{F}_{jk}\hat{F}_{ki}) = 1 = \exp 2\pi i(c_1)_{ij}$$

implies $F_{ij}F_{jk}F_{ki} = c_{ijk}$, cocycle representing $\beta(\Gamma) \in H^2(\Lambda^\times_{\text{ev}})$.

## 4. In progress

Action of odd flows on $N = 2$ differential operators.

Action of $N = 2$ super Virasoro algebra as additional symmetries.

Choice of trivialization removing constant term from $N = 1$ Lax operator $\underline{L}$.