SUPERSTRING SCATTERING AMPLITUDES

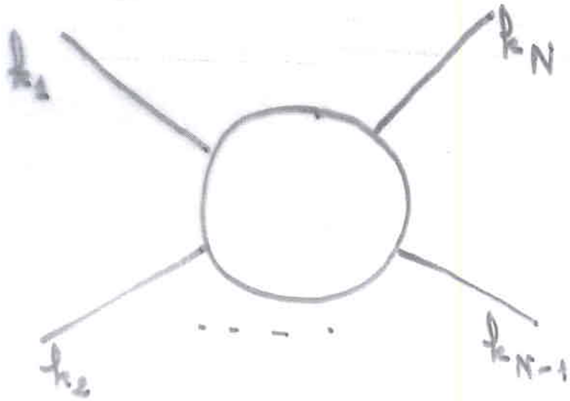
AND

MODULAR FORMS

JOINT WORK WITH

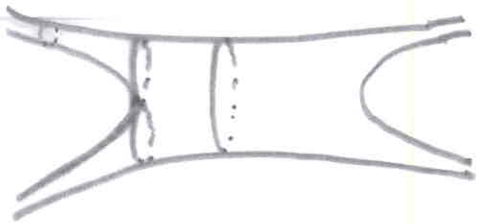
ERIC D'HOKER

SCATTERING AMPLITUDES



$$= \mathbb{A}(k_1, \ldots, k_N)$$

STRING THEORY



$$\delta(k) \prod_{i=1}^{4} \varepsilon^{\mu_i \bar{\mu}_i} K \bar{K} \; \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})}$$

$(s = k_1 \cdot k_2 \quad t = k_2 \cdot k_3 \quad u = k_3 \cdot k_1)$

Virasoro - Shapiro Formula

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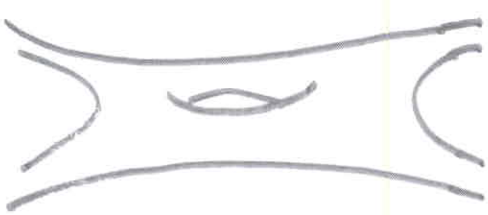

Type II: Green - Schwarz (1982)

Heterotic: Gross et al. (1985)

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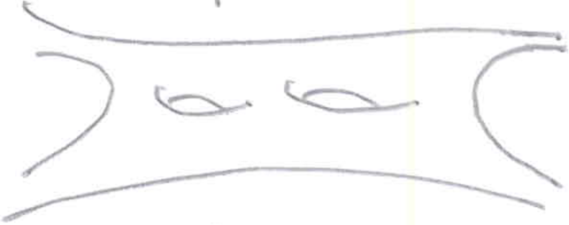

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Geometric Difficulty: "SUPERMODULI"

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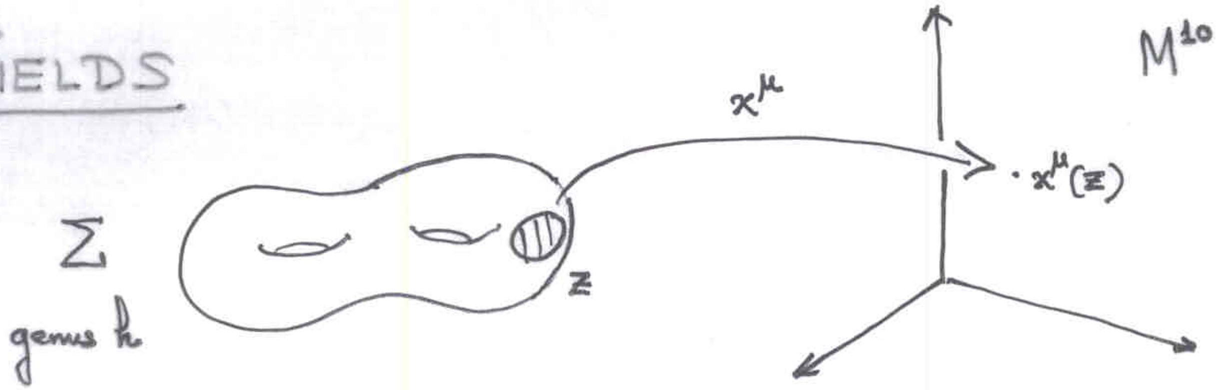
⋮

THE RNS FORMULATION

FIELDS



$$x^\mu(z)\,,\quad \psi_+^\mu(z), \psi_-^\mu(z) \longrightarrow X^\mu(z,\theta,\bar\theta) = x^\mu + \theta\psi_+^\mu + \bar\theta\psi_-^\mu$$

$$g_{mn}(z),\quad \chi_{\bar z}^{\ +}, \chi_z^{\ -} \longrightarrow \begin{cases} dz^M E_M{}^A = E^A \\ E_m{}^a = e_m{}^a + \theta\gamma^a\chi_m \end{cases}$$

"2-dimensional supergeometry"

$$\begin{aligned}\psi_+^\mu &= \psi_+^\mu(z)(dz)^{1/2} \\ \chi_{\bar z}^{\ +} &= \chi_{\bar z}^{\ +}\,(d\bar z)\otimes(dz)^{-1/2}\end{aligned} \longrightarrow \boxed{\text{spin structure } \delta}$$

ACTION

$$\begin{aligned} I_m &= \frac{1}{4\pi}\int d^2 z\, \Big(\partial_z x^\mu \partial_{\bar z} x^\mu - \psi_+^\mu \partial_{\bar z}\psi_+^\mu - \psi_-^\mu \partial_z \psi_-^\mu \\ &\qquad + \chi_{\bar z}^{\ +}\psi_+^\mu \partial_z x^\mu + \chi_z^{\ -}\psi_-^\mu \partial_{\bar z} x^\mu - \tfrac{1}{2}\chi_{\bar z}^{\ +}\chi_z^{\ -}\psi_+^\mu\psi_-^\mu \Big) \\ &= \frac{1}{4\pi}\int d^{2|2} z\, (\operatorname{sdet} E_M{}^A)\, \mathcal{D}_+ X^\mu \mathcal{D}_- X^\mu \end{aligned}$$

SYMMETRIES

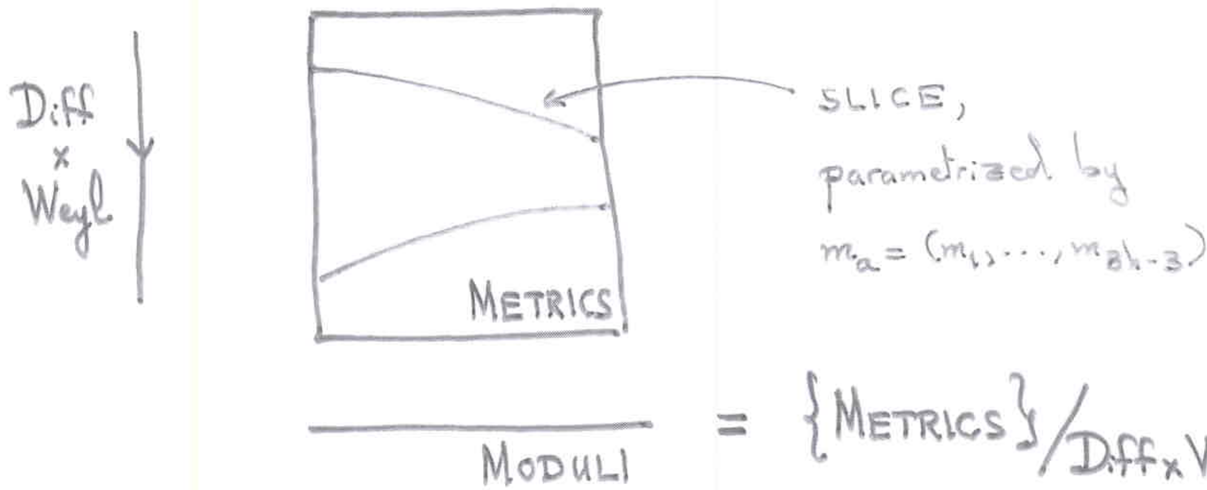
$$\begin{aligned} \mathrm{Diff}(\Sigma) &: \quad \delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m \\ \text{susy} &: \quad \delta\chi_{\bar z}^{\ +} = -2\,\nabla_{\bar z}\,\delta\zeta^+ \end{aligned} \longrightarrow \begin{cases} \mathrm{sDiff}(\Sigma) \\ \delta V^m = \delta v^m - \theta\gamma^m\delta\zeta \end{cases}$$

"super-reparametrizations"

Construction of Superstring Amplitudes

I) Non-chiral Amplitudes

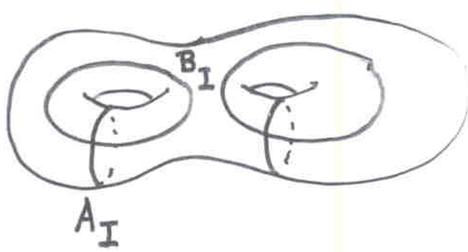
$$\mathbb{A}^{nc}[\delta] = \int D(g_{mn}\,\bar{\chi}\,\chi_{\bar z}^{+}\,x^\mu\,\psi_{\pm}^{\mu})\; e^{-I_m}$$



$$\text{Moduli} = \{\text{Metrics}\}/\text{Diff} \times \text{Weyl}$$

$$\mathbb{A}^{nc}[\delta] = \int_{\text{SLICE}} \mathcal{A}(m,\bar m)\,(dm_1 \wedge \cdots \wedge dm_{3h-3}) \otimes (d\bar m_1 \wedge \cdots \wedge d\bar m_{3h-3})$$

Period Matrix:



$$\oint_{A_I} \omega_J = \delta_{IJ}\,, \qquad \oint_{B_I} \omega_J = \Omega_{IJ}$$

$$\omega_J = \{\text{holomorphic 1-forms}\}$$

II) Chiral Splitting

$$\mathbb{A}^{nc}[\delta] = \int (\det \operatorname{Im}\Omega)^{-5}\; d\mu[\delta](\Omega) \wedge \overline{d\mu[\delta](\Omega)}$$

III) GSO Projection

$$\boxed{\mathbb{A} = \int (\det \operatorname{Im}\Omega)^{-5} \Big\{ \sum_\delta \eta_\delta\, d\mu[\delta](\Omega) \Big\} \wedge \overline{\Big\{ \sum_\delta \eta_\delta\, d\mu[\delta](\Omega) \Big\}}}$$

THE PICTURE-CHANGING ANSATZ (Friedan, Martinec, Shenker 1985)

$x^\mu, \psi_+^\mu, \psi_-^\mu$ "matter fields"

$b, c, \beta, \gamma, \bar{b}, \bar{c}, \bar{\beta}, \bar{\gamma}$ "ghost fields"

$$\boxed{d\mu[\delta] = \prod_{a=1}^{3h-3} dm_a \left\langle \prod_{a=1}^{3h-3} \langle \mu_a | b \rangle \prod_{\alpha=1}^{2h-2} Y(z_\alpha) \right\rangle}$$

$Y(z) = \delta(\beta(z))\, S(z)$ "picture changing operator"

$S(z) =$ "supercurrent"

$$= -\frac{1}{2}\psi_+^\mu \partial_z x^\mu + \frac{1}{2} b\gamma - \frac{3}{2}\beta\partial_z c - (\partial_z \beta) c$$

$\{z_\alpha\} = \{2h-2 \text{ arbitrary points}\}$

$m_a = (3h-3)$ moduli parameters for a gauge slice

$\mu_a =$ corresponding Beltrami differentials

AMBIGUITIES OF THE PICTURE-CHANGING ANSATZ

$$\boxed{\delta(d\mu[\delta](z_\alpha)) = \text{Exact Differential}}$$

(Verlinde & Verlinde, 87)

HOWEVER, THESE DIFFERENTIALS ARE ONLY DEFINED <u>LOCALLY</u>!

IDEAS AND DEVELOPMENTS

- Descent Equations (H. Verlinde, 87)
- Assuming global sections and studying boundary terms (Atick, Moore, Sen, 88)
- Unitary gauge $\begin{cases} z_\alpha \text{ given by } \omega(z_\alpha) = 0 \\ \omega(z) \text{ hol. 1-form} \end{cases}$

  (Lechtenfeld, Parkes, Morozov, Iengo et al. 88–90)
- Light-cone gauge (Mandelstam 84–90)
- Operator Methods (Alvarez-Gaumé et al. 88
  Neveu and West 87–89
  Di Vecchia, Sciuto, et al. 87–89)
- Algebraic Geometry and structure of modular forms (Catenacci, et al.
  Moore, Harris, Nelson, and Singer, 86)
- Supermanifolds and super algebraic geometry (Baranov and A. Schwarz,
  Beilinson and Manin, Rabin
  Atick, Rabin, and Sen 86)
- Fischler-Susskind mechanism (La and Nelson, 88)

MAIN RESULTS

GENUS $h=2$

$$\boxed{d\mu[\delta](\Omega) = \frac{1}{16\pi^6} \; \frac{\Xi_6[\delta](\Omega)\;\; \vartheta[\delta]^4(\Omega)}{\Psi_{10}(\Omega)} \prod_{I\leq J} d\Omega_{IJ}}$$

- $$\Psi_{10}(\Omega) = \prod_{\kappa \text{ even}} \vartheta[\kappa](\Omega)^2$$: basic modular form of weight 10

$$\Psi_{10}\left(\frac{A\Omega+B}{C\Omega+D}\right) = \det(C\Omega+D)^{10}\, \Psi_{10}(\Omega)$$

- Genus $h=2$: 10 even spin structures δ
  6 odd spin structures ν

$$\delta \text{ even} \iff \delta = \nu_1+\nu_2+\nu_3$$

$$\langle \nu_i | \nu_j \rangle = \exp\left(4\pi i\,(\nu_i' \nu_j'' - \nu_i'' \nu_j')\right) \qquad \nu_i = (\nu_i' | \nu_i'')$$

$$\boxed{\Xi_6[\delta](\Omega) = \sum_{1\leq i<j\leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i+\nu_j+\nu_k]^4(\Omega)}$$

• Absence of Ambiguities

$$\boxed{d\mu[\delta](\Omega) = \frac{\prod_{I\le J} d\Omega_{IJ}}{\det \omega_I\omega_J(p_a)} \left\langle \prod_{a=1}^{3} b(p_a) \prod_{\alpha=1}^{2} \delta(\beta(q_\alpha)) \right\rangle \sum_{k=1}^{6} \mathcal{X}_k}$$

$$\begin{aligned}\mathcal{X}_1+\mathcal{X}_6 = \frac{1}{16\pi^2}\Big\{ & -10\, S_\delta(q_1,q_2)\,\partial_{q_1}\partial_{q_2}\ln E(q_1,q_2) \\ & -\partial_{q_1}G_2(q_1,q_2)\,\partial\psi_1^*(q_2) + \partial_{q_2}G_2(q_2,q_1)\,\partial\psi_1^*(q_1) \\ & +2\,G_2(q_1,q_2)\,\partial\psi_1^*(q_2)\, f^{(1)}_{3/2}(q_2) - 2\,G_2(q_2,q_1)\,\partial\psi_2^*(q_1)\, f^{(2)}_{3/2}(q_1)\Big\}\end{aligned}$$

$$\mathcal{X}_2 = \frac{1}{16\pi^2}\,\omega_I(q_1)\,\omega_J(q_2)\,S_\delta(q_1,q_2)\left[\partial_I\partial_J \ln\frac{\theta[\delta](0)^8}{\theta[\delta](D_\beta)} + \partial_I\partial_J \ln\theta[\delta](D_b)\right]$$

$\mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5$ = similar explicit formulas

$S_\delta(q_1,q_2)$: Szegö kernel; G_2: Green's function

D_β, D_b: divisors of q_α and p_b

. The Right Hand Side is independent of p_b and q_α

. $\mathcal{X}_1 = -\frac{1}{8\pi^2}\int d^2z\, d^2w\; \chi(z)\chi(w)\langle S(z)S(w)\rangle$

$\mathcal{X}_k$, $2\le k\le 6$: global correction terms

$S(z)$ = supercurrent $= -\frac{1}{2}\psi_+\partial_z x + \cdots$

- MODULAR INVARIANCE AND GSO

$$\begin{pmatrix} \tilde{\vec\delta}' \\ \tilde{\vec\delta}'' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \vec\delta' \\ \vec\delta'' \end{pmatrix} + \frac{1}{2}\,\mathrm{diag}\begin{pmatrix} CD^T \\ AB^T \end{pmatrix} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4,\mathbb{Z})$$

$$\tilde\Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

$$\det \mathrm{Im}\,\tilde\Omega = |\det(C\Omega+D)|^{-2} \det \mathrm{Im}\,\Omega$$

$$\Theta[\tilde\delta](\tilde\Omega)^4 = \varepsilon^4 \det(C\Omega+D)^2\, \Theta[\delta](\Omega)^4 \qquad \varepsilon^8 = 1$$

$$\Psi_{10}(\tilde\Omega) = \det(C\Omega+D)^{10}\, \Psi_{10}(\Omega)$$

$$\boxed{\Xi_6[\tilde\delta](\tilde\Omega) = \varepsilon^4 \det(C\Omega+D)^6\, \Xi_6[\delta](\Omega)}$$

$$\prod_{I\le J} d\tilde\Omega_{IJ} = \det(C\Omega+D)^{-3} \prod_{I\le J} d\Omega_{IJ}$$

$$\boxed{d\mu[\tilde\delta](\tilde\Omega) = \det(C\Omega+D)^{-5}\, d\mu[\delta](\Omega)}$$

$$\boxed{d\mu(\Omega) = \sum_\delta d\mu[\delta](\Omega) \quad \text{IS THE UNIQUE MODULAR INVARIANT GSO PROJECTION}}$$

- VANISHING OF THE COSMOLOGICAL CONSTANT

$$\boxed{\begin{array}{c} \sum_\delta \Xi_6[\delta](\Omega)\, \vartheta^4[\delta](\Omega) = 0 \quad \text{along divisor of nodes} \\ \Downarrow \ \text{J.I. IGUSA'S THEOREM} \\ \sum_\delta \Xi_6[\delta](\Omega)\, \vartheta^4[\delta](\Omega) \equiv 0 \\ \Downarrow \\ d\mu(\Omega) \equiv 0 \quad \text{for every } \Omega \end{array}}$$

c.f. RIEMANN IDENTITIES

$$\sum_\delta \langle \nu|\delta\rangle\, \vartheta[\delta]^4(\Omega) \equiv 0 \quad \text{for each } \nu \text{ odd spin structure}$$

$\Longrightarrow$ Modular Invariant combinations are <u>not</u> unique!
(GSO "projections")

- VANISHING OF THE 1,2,3 POINT FUNCTIONS

$$\sum_\delta \Xi_6[\delta](\Omega)\, \vartheta[\delta](\Omega)^4\, S_\delta(z_1,z_2)^2 = 0$$

$$\sum_\delta \Xi[\delta](\Omega)\, \vartheta[\delta](\Omega)^4\, S_\delta(z_1,z_2)\, S_\delta(z_2,z_3)\, S_\delta(z_3,z_1) = 0$$

- THE 4-POINT FUNCTION

Contribution of the disconnected diagrams

$$\langle S(z)S(w)\rangle \langle \prod V(k_i,\varepsilon_i)\rangle \quad , \quad \langle T(z)\rangle \langle \prod V(k_i,\varepsilon_i)\rangle$$

$$\boxed{\begin{aligned}\langle \prod_{i=1}^{4} V(k_i,\varepsilon_i)\rangle_{disc} &= \delta(k) \int (\det \operatorname{Im}\Omega)^{-5} \Big| \prod_{I\leq J} d\Omega_{IJ} \Big|^2 \\ &\times \prod_{i=1}^{4} \int_{\Sigma^4} d^2 z_i \, |\mathcal{F}|^2 \exp\Big(-\sum_{i<j} k_i \cdot k_j \, G(z_i,z_j)\Big)\end{aligned}}$$

$$\mathcal{F} = K_S \, S(1234) + \sum_{ijkl} K_T(ij|kl) \, T(ij|kl)$$

- $$S(1234) = \omega_I(z_1)\omega_J(z_2)\omega_K(z_3)\omega_L(z_4) \times \sum_{\delta} \frac{\Xi_6[\delta]\, \Theta[\delta]^3 \, \partial_I \partial_J \partial_K \partial_L \Theta[\delta](0,\Omega)}{\Psi_{10}(\Omega)} \left(-\frac{1}{192\pi}\right.$$

- $$T(ij|kl) = -\frac{1}{8\pi^2}\, \omega_{[1}(z_i)\,\omega_{2]}(z_j)\, \omega_{[1}(z_k)\,\omega_{2]}(z_l)$$

- $\langle \prod_{i=1}^{4} V(k_i,\varepsilon_i) \rangle_{disc}$ is finite, in the regime of purely imaginary $s_{ij} = k_i \cdot k_j$

(the other regimes are accessible by analytic continuation)

$$\sum \Xi_6[\delta]\, \Theta[\delta]^3\, \partial_I \partial_J \partial_K \partial_L \Theta[\delta](0,\Omega) \text{ vanishes of}$$

second order along the divisor of nodes

- K_S : same kinematic factor encountered at tree and 1-loop level

→ same ttR^4 corrections to Einstein's action

(Gross-Witten, 1986)

- $K_T(ij|kl)$: new kinematic factor, specific to two-loop

→ new R^4 corrections to Einstein's action

$$(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu})^2 - R_{\alpha\beta\mu\nu}R^{\gamma\delta\mu\nu}R^{\alpha\beta\rho\sigma}R_{\gamma\delta\rho\sigma}$$
$$+ 4\, R_{\alpha\beta\mu\nu}R^{\gamma\delta\mu\nu}R^{\beta}{}_{\gamma\rho\sigma}R_{\delta}{}^{\alpha\rho\sigma} - 4\, R^{\alpha\beta\mu\nu}R_{\delta\alpha\mu\nu}R_{\beta\gamma\rho\sigma}R^{\gamma\delta\rho\sigma}$$
$$+ 4\, R^{\alpha\beta\mu\nu}R_{\beta\gamma\nu\rho}R^{\gamma\delta\rho\sigma}R_{\delta\alpha\sigma\mu} - 4\, R^{\alpha\beta\mu\nu}R_{\beta\delta\nu\rho}R_{\delta\alpha}{}^{\rho\sigma}R^{\gamma\delta}{}_{\sigma\mu}$$

- but $\langle \prod_{i=1}^{4} V(k_i,\varepsilon_i) \rangle_{conn} \longrightarrow$?

• Compactifications to a Manifold/Orbifold C

- Compactification modifies only the matter part, not the ghost part
- Worldsheet $N=1$ local supersymmetry is preserved

Z_C = chiral partition function for the matter part of C

Z_M = " " of Minkowski space

Then the superstring measure is independent of any choice of gauge slice, and

$$\boxed{\mathcal{A}_C[\delta] = \frac{Z_C}{Z_M}\left\{ \mathcal{Z} + \frac{\zeta_1\zeta_2}{16\pi^6} \frac{\Xi_6[\delta]\,\vartheta[\delta]^4}{\Psi_{10}} - \frac{\zeta_1\zeta_2}{4\pi^2}\,\mathcal{Z}\langle S_C(q_1) S_C(q_2)\rangle \right\}}$$

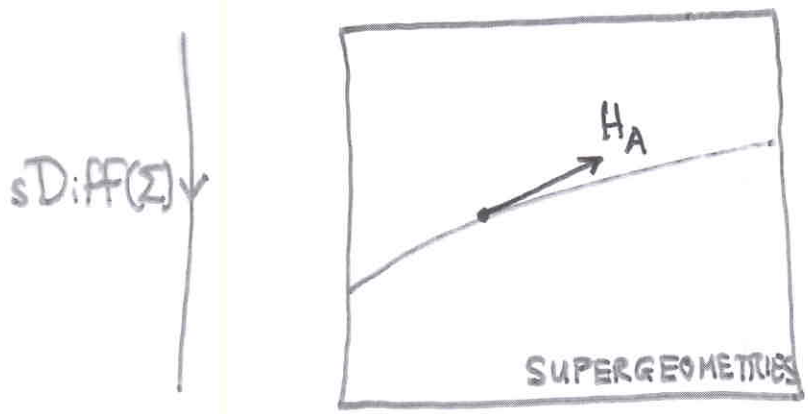
in the gauge $S_\delta(q_1,q_2)=0$. Here

$S_C(q)$ = supercurrent

$$\mathcal{Z} = \frac{1}{\det \omega_I\omega_J(p_a)} \left\langle \prod_{a=1}^{3} b(p_a) \prod_{\alpha=1}^{2} \delta(\beta(q_\alpha)) \right\rangle$$

GAUGE-FIXING IN THE SUPERFIELD FORMALISM

$$\mathcal{A}^{nc}[\delta] = \int D(g_{mn}\,\chi_{\alpha}\,\psi^{\mu}_{\pm})\, e^{-I_m} = \int DE_M{}^A\, DX^{\mu}\, e^{-I_m(E_M{}^A,\, X^{\mu})}$$



slice

H_A = super Beltrami differentials

SUPERMODULI $= \{E_M{}^A\}/_{\text{sDiff} \times \text{sWeyl}}$

- $$\dim(\text{Supermoduli}) = \left(\dim \operatorname{Coker} \bar\partial_{(-1)} \,\middle|\, \dim \operatorname{Coker} \bar\partial_{(-\frac{1}{2})}\right)$$
  $$= \left(\dim \operatorname{Ker} \bar\partial_{2} \,\middle|\, \dim \operatorname{Ker} \bar\partial_{3/2}\right)$$
  $$= (3h-3 \mid \underbrace{2h-2}) \qquad \underline{\text{odd supermoduli}}$$
- Super Beltrami differentials
  $$H \equiv H_-{}^{z} = E_-{}^{M}\, \delta E_M{}^{z}$$
  $$H_A = E_-{}^{M}\, \frac{\partial E_M{}^{z}}{\partial m^A} \qquad m^A = (m^a \mid m^\alpha) \text{ parameters for the supermoduli slice}$$
- $$\boxed{H = \bar\theta\left(\delta\mu_{\bar z}{}^{z} - \theta\,\delta\chi_{\bar z}{}^{+}\right)}$$

Reliable Gauge-Fixing in the Superfield Formalism

$$\mathbb{A}^{nc}[\delta] = \int_{\text{SLICE}} \Big| \prod_{A=1}^{(3h-3|2h-2)} dm^A \Big|^2 \int D(B\bar{B}C\bar{C}X^\mu)\, e^{-I_m - I_{gh}}$$

$$\times \prod_{A=1}^{(3h-3|2h-2)} \big| \delta(\langle H_A | B \rangle) \big|^2$$

$$\boxed{\mathbb{A}^{nc}[\delta] \equiv \int_{\text{SLICE}} \Big| \prod_{A=1}^{(3h-3|2h-2)} dm^A \Big|^2 \Big\langle \prod_{A=1}^{(3h-3|2h-2)} \big| \delta(\langle H_A | B \rangle) \big|^2 \Big\rangle}$$

$$I_m = \frac{1}{4\pi} \int d^{2|2} z\, (\operatorname{sdet} E_M{}^A)\, \mathcal{D}_+ X^\mu \mathcal{D}_- X^\mu$$

$$I_{sgh} = \frac{1}{2\pi} \int d^{2|2} z\, (\operatorname{sdet} E_M{}^A)\, (B \mathcal{D}_- C + \bar{B} \mathcal{D}_+ \bar{C})$$

$$B = \beta(z) + \theta\, b(z)$$

$$C = c(z) + \theta\, \gamma(z)$$

"Superghosts"

$b = b(z)(dz)^2$; $\beta = \beta(z)(dz)^{3/2}$

$c = c(z)(dz)^{-1}$; $\gamma = \gamma(z)(dz)^{-1/2}$

Main Difficulty in Superstring Perturbation Theory

- Gauge-fixing produces an integral over Supermoduli Space
- Have to integrate out <u>odd</u> supermoduli m^α to descend to Moduli Space

GAUGE-FIXED FORMULA IN COMPONENTS

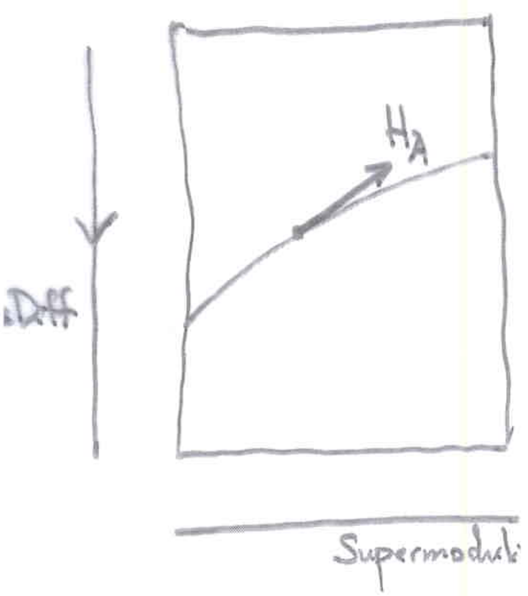
$$I_m + I_{sgh} = I_{free} - \frac{1}{2\pi}\int \chi_{\bar z}{}^{+} S(z) - \frac{1}{2\pi}\int \chi_z{}^{-} \bar S(\bar z) - \frac{1}{2}\chi\bar\chi\psi_+\psi_-$$

$$I_{free} = \frac{1}{4\pi}\int d^2z \,\big(\partial_z x^\mu \partial_{\bar z} x^\mu - \psi_+ \partial_{\bar z}\psi_+ - \psi_- \partial_z \psi_- + b\,\partial_{\bar z} c + \beta\,\partial_{\bar z}\gamma + \bar b\,\partial_z \bar c + \bar\beta\,\partial_z \bar\gamma \big)$$

$$S(z) = -\frac{1}{2}\psi_+^\mu \partial_z x^\mu + \frac{1}{2} b\gamma - \frac{3}{2}\beta\partial_z c - (\partial_z\beta)c$$

"supercurrent"

$$\boxed{A^{nc}[\delta] = \int_{\text{SLICE}} \left|\prod dm^A\right|^2 \left\langle \left|\prod_A \delta(\langle H_A|B\rangle)\right|^2 e^{-\int\chi S - \int\bar\chi\bar S + \frac{1}{2}\int\chi\bar\chi\psi_+\psi_-} \right\rangle}$$



$m^A = (m^a | \zeta^\alpha)$ supermoduli parameters

$m^A \to (g(m^A), \chi(m^A))$ slice

$$\boxed{\begin{aligned} H_A &= \bar\Theta\,(\mu_A - \Theta\,\nu_A) \\ \mu_A &= \tfrac{1}{2} g^{z\bar z}\,\delta_A g_{\bar z\bar z}\,, \quad \nu_A = \frac{\partial\chi}{\partial m^A} \end{aligned}}$$

$$\boxed{\delta(\langle H_A|B\rangle) = \delta(\langle\mu_A|b\rangle - \langle\nu_A|\beta\rangle)}$$

Naive Derivation of the Picture-Changing Ansatz

(E. Verlinde and H. Verlinde, 1987)

Gauge Slice: $g_{mn}(\Omega)$, $\chi = \sum_{\alpha=1}^{2h-2} \zeta_\alpha \chi_\alpha$ $\qquad m^A = (\Omega_{IJ} | \zeta_\alpha)$

$$H_a = \bar{\Theta}\, \mu_a \quad , \quad H_\alpha = -\bar{\Theta}\Theta\, \chi_\alpha$$

Naive Chiral Splitting: drop $\chi \bar{\chi} \psi_+ \psi_-$

$$d\mu[\delta] = \prod_a dm^a \int \prod d\zeta_\alpha \int D(x b c \beta \gamma)\, e^{-I + \int \chi S} \prod_{a=1}^{3h-3} (\mu_a | b) \times \prod_{\alpha=1}^{2h-2} \delta(\langle \chi_\alpha | \beta \rangle)$$

$$= \prod_a dm^a \langle \prod_{a=1}^{3h-3} (\mu_a | b) \prod_{\alpha=1}^{2h-2} \langle \chi_\alpha | S \rangle\, \delta(\langle \chi_\alpha | \beta \rangle) \rangle$$

$$\chi_\alpha = \delta(z, z_\alpha)$$

$$d\mu[\delta] = \prod_a dm^a \langle \prod_{a=1}^{3h-3} (\mu_a | b) \prod_{\alpha=1}^{2h-2} Y(z_\alpha) \rangle, \quad Y(z) = S(z)\, \delta(\beta(z))$$

$=$ Picture-Changing Ansatz !?

ORIGIN OF AMBIGUITIES

$$\begin{array}{ccc} (g_{mn}, \chi_m^{\alpha}) & \xrightarrow{\text{SUSY}} & (\tilde{g}_{mn}, \tilde{\chi}_m^{\alpha}) \\ \downarrow & & \downarrow \\ g_{mn} & \xrightarrow{\text{Diff} \times \text{Weyl}} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!/ \quad & \tilde{g}_{mn} \end{array}$$

$$\begin{array}{ccc} [(g_{mn}, \chi_m^{\alpha})] & \in & \text{Supermoduli} \\ \not\downarrow & & \\ [g_{mn}] & \in & \text{Moduli} \end{array}$$

MAIN PROPOSAL (D'Hoker & P., 1988)

$$\boxed{\begin{array}{c} (g_{mn}, \chi_m^{\alpha}) \\ \downarrow \\ \hat{\Omega}_{IJ} \end{array}} \qquad \delta \hat{\Omega}_{IJ} = 0 \text{ under SUSY transformations}$$

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int d^2z \int d^2w \, \omega_I(z) \chi_{\bar{z}}^{+} S(z,w) \chi_{\bar{w}}^{+} \omega_J(w)$$

$$\boxed{\text{Integrating Odd supermoduli} = \text{Integrating over the fibers of this projection}}$$

CHIRAL SPLITTING

Obstructions to separate left from right movers

- $\chi\bar{\chi}\psi_+\psi_-$ terms ; $\mathcal{D}_+, \mathcal{D}_-$ covariant derivatives
- $\langle x(z)x(w)\rangle = G(z,w)$, $-\partial_{\bar z}\partial_z G(z,w) = 2\pi\delta(z,w) - 2\pi \frac{1}{\int d^2z\sqrt{g}}$

Effective Rules (D'Hoker & P., 1988)

$$\left\{\begin{array}{lcl} x(z) & \longrightarrow & x_+(z) \\ \langle x(z)x(w)\rangle & \longrightarrow & \langle x_+(z)x_+(w)\rangle = -\ln E(z,w) \\ & & \text{"prime form"} \\ \mathcal{D}_+ & \longrightarrow & \partial_\theta + \theta\partial_z \end{array}\right.$$

with insertion of $\exp\left(p_I^\mu \oint_{B_I} dz\, \partial_z x_+^\mu\right)$

p_I^μ = "internal loop momenta" (Verlinde-Verlinde, bosonic string, 87)

$$\boxed{\begin{aligned} A[\delta] &= \int |\prod_A dm^A|^2 \int dp_I^\mu \,|e^{i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu} \mathcal{A}[\delta]|^2 \\ &= \int (\det \operatorname{Im}\hat{\Omega})^{-5} |\prod_A dm^A \mathcal{A}[\delta]|^2 \end{aligned}}$$

$$\mathcal{A}[\delta] = \langle \prod_A \delta(\langle H_A|B\rangle) \exp\left(\int \frac{d^2z}{2\pi} \chi(z) S(z)\right)\rangle$$

COMPLICATIONS AND PAY-OFF

Supermoduli Parameters: $m^A = (\hat{\Omega}_{IJ} | \zeta^\alpha)$

- Beltrami sdifferentials H_A no longer split, e.g.,

$$\delta\hat{\Omega}_{IJ} = 0 \implies \begin{cases} H_A = \bar{\theta}(\delta\mu_A - \theta\,\delta\chi_A) \\ \delta\mu_A \neq 0 \ \underline{\text{and}} \ \delta\chi_A \neq 0 \end{cases}$$

$$\boxed{\prod_{A=1}^{(3|2)} \delta(\langle H_A | B \rangle) \neq \prod_{a=1}^{3} (\delta\mu_a | b) \prod_{\alpha=1}^{2} (\delta\chi_\alpha | \beta)}$$

- $\hat{\mu}$- slice independence

Worldsheet correlation functions require a metric $\hat{g}_{mn}$, $[\hat{g}_{mn}] = \hat{\Omega}_{IJ}$

$$\begin{array}{ccc} & & (g_{mn}, \chi_m{}^\alpha) \\ & & \downarrow \\ g_{mn} + \underbrace{\delta g_{mn}}_{\hat{\mu}} = \hat{g}_{mn} & \xleftarrow{\text{choice}} & \hat{\Omega}_{IJ} \end{array}$$

The final answer has to be independent of the choice of $\hat{\mu}$!

- **Stress Tensor Corrections**

Deformations of Complex Structures

$$g_{mn} \longrightarrow \hat{g}_{mn} = g_{mn} + \delta g_{mn}$$

$$\partial_{z^*} \longrightarrow \partial_{\hat{z}^*} = \partial_{z^*} + \hat{\mu}\,\partial_z$$

$$\boxed{\langle \cdots \rangle_{g_{mn}} \longrightarrow \langle \cdots \rangle_{\hat{g}_{mn}} + \int d^2z\, \hat{\mu} \langle T(z) \cdots \rangle}$$

- **Superholomorphicity and Holomorphicity**

$$\left\{ \begin{array}{c} \mathscr{L}_{-} \hat{\omega}_I = 0 \\ \text{w/r to the supergeometry } (g_{mn}, \chi) \\ E_m{}^a = e_m{}^a + \theta \gamma^a \chi_m \end{array} \right\} \longleftrightarrow \boxed{\begin{array}{c} \hat{\omega}_I = \theta\, \omega_I(z) + \lambda_I(z) \\ \omega_I \text{ hol\'c w/r to } \hat{\Omega}_{IJ} \end{array}}$$

$$\left\{ \begin{array}{rl} d\hat{\Omega}_{IJ} = & \Phi_{IJ} \quad 3/2\text{-diffs} \\ = & \hat{\omega}_I \mathscr{D}_+ \hat{\omega}_J \end{array} \right\}$$

$$\prod_A \delta(\langle H_A | B\rangle) = \frac{\operatorname{sdet}\langle H_A | \Phi_C\rangle}{\operatorname{sdet}\langle H^*_A | \Phi_C\rangle} \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))$$

$$H^*_a = \bar\theta\, \delta(z, p_a) \quad , \quad H^*_\alpha = \bar\theta \theta\, \delta(z, q_\alpha)$$

$$\Downarrow$$

$$\mathcal{A}[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det \Phi_{IJ+}(p_a) \det\langle H_\alpha | \Phi^*_\beta\rangle}$$

$$\times \left\{ 1 - \frac{1}{8\pi^2} \int d^2_z\, \chi_{\bar z}{}^+ \int d^2_w\, \chi_{\bar w}{}^+ \langle S(z) S(w)\rangle \right.$$

$$\left. + \frac{1}{2\pi} \int d^2_z\, \hat\mu_{\bar z}{}^z \langle T(z)\rangle \right\}$$

$$\Downarrow$$

$$\mathcal{A}[\delta] = i\, \frac{\langle \prod b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det \omega_I \omega_J(p_a) \det\langle \chi_\alpha | \psi^*_\beta\rangle} \left\{ 1 + \sum_{j=1}^{6} X_j \right\}$$

$$\Downarrow$$

Set $\chi_\alpha = \delta(z, x_\alpha)$ and let $x_\alpha \to q_\alpha$.

EVALUATION IN TERMS OF Θ-CONSTANTS

- Choose the split gauge

$$\boxed{S_\delta(q_1, q_2) = 0}$$

(e.g. q_1 is arbitrary, and there are two corresponding choices for q_2.)

- $$\mathcal{M}_{\nu_i\nu_j}(\Omega) = \partial_1\Theta[\nu_i](\Omega,0)\,\partial_2\Theta[\nu_j](\Omega,0) - \partial_2\Theta[\nu_i](\Omega,0)\,\partial_1\Theta[\nu_j](\Omega,0)$$

$$d\mu[\delta](\Omega) = \prod_{I\leq J} d\Omega_{IJ}\ \frac{\langle\nu_1|\nu_2\rangle\mathcal{M}^4_{\nu_1\nu_2} + \langle\nu_2|\nu_3\rangle\mathcal{M}^4_{\nu_2\nu_3} + \langle\nu_3|\nu_1\rangle\mathcal{M}^4_{\nu_3\nu_1}}{16\pi^2\mathcal{M}^2_{\nu_1\nu_2}\mathcal{M}^2_{\nu_2\nu_3}\mathcal{M}^2_{\nu_3\nu_1}}\,\Theta^4[\delta]$$

- $$\mathcal{M}_{\nu_i\nu_j} = \pm\pi^2 \prod_{k\neq i,j} \Theta[\nu_i+\nu_j+\nu_k](\Omega)$$

- $$\sum_\delta \Xi_6[\delta](\Omega)\,\Theta^4[\delta](\Omega) = 2\sum_\delta \Theta^{16}[\delta] - \frac{1}{2}\Big(\sum_\delta \Theta^8[\delta]\Big)^2$$

$$= 0$$