Unique Factorization of Feynman Graphs

D. Kreimer

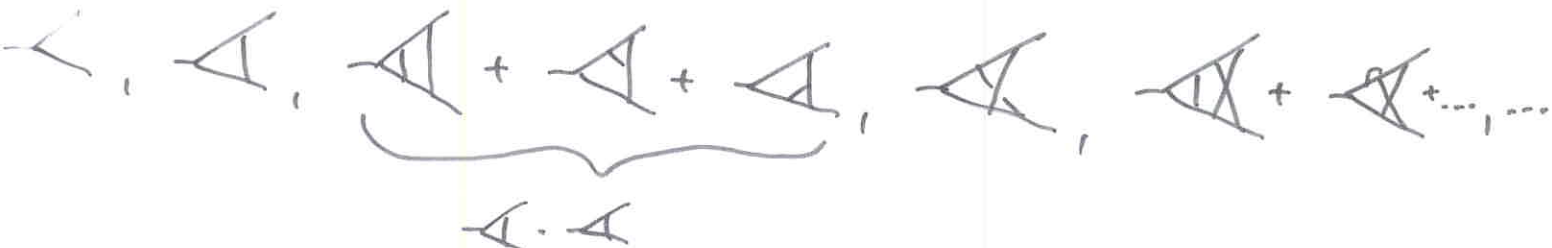
- (pre-) Lie algebras of graphs
- Hopf algebras
- Renormalization
- "prime graphs"
- factorization of D-S eq.
- gauge theories

collab. with D.J. Broadhurst, A. Connes

We can count.

1, 2, 3, 2·2, 5, 2·3, 7, 2·2·2, 3·3, 2·5, 11, ..

perturbative expansion



"prime decomposition" ??

What about the "primes" itself?

To what extent do they determine the general graph?

How unique is the decomposition of a problem into subproblems?

How can I efficiently reassemble the total from its parts?

set-up

- Lie algebra of insertions of graphs
- Hopf algebra of decomposition
- decompose problem until its simple
- evaluate $\hat{=}$ Feynman rules
- group structure of characters on H
- compare!
  - choose renormalization scheme

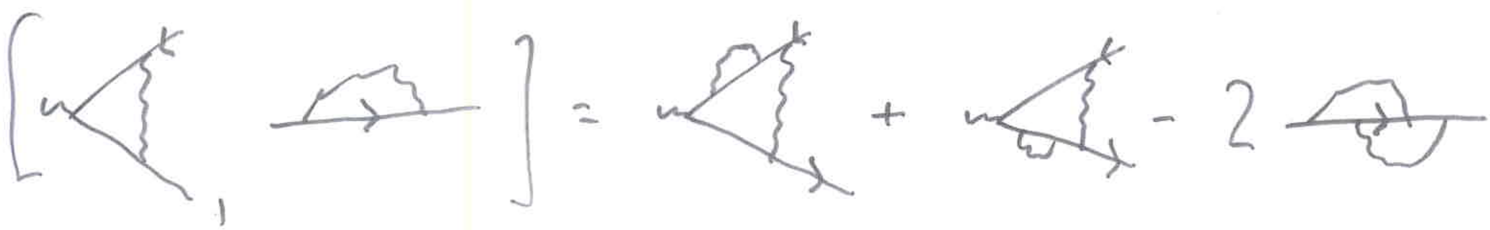
"$S_R^\phi * \phi$"

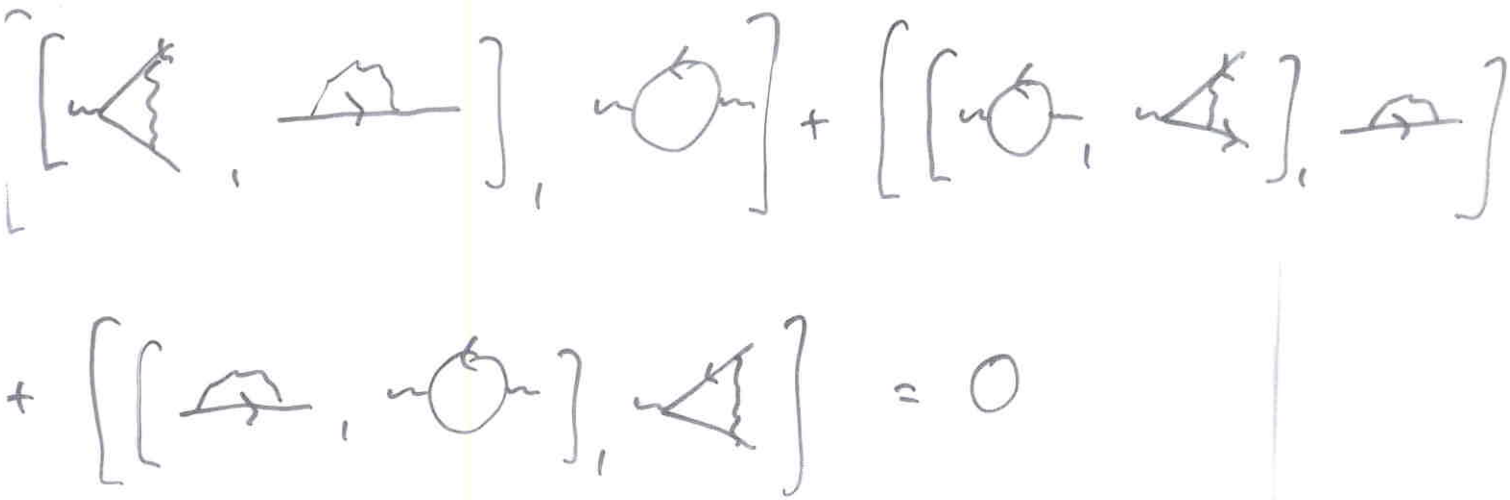
$\bar{\phi}^{-1} * \phi$ $\qquad\qquad$ $\bar{\phi} = S_R^\phi \circ S$

The Lie algebra of Feynman graphs

$\Gamma_1 =$  $\Gamma_2 =$  $\Gamma_3 =$  [QED, say]

$[\Gamma_1, \Gamma_2] =$ "2 into 1 − 1 into 2"





Jacobi: $\sum_{\text{cycl}} [[\Gamma_i, \Gamma_j], \Gamma_k] = 0$

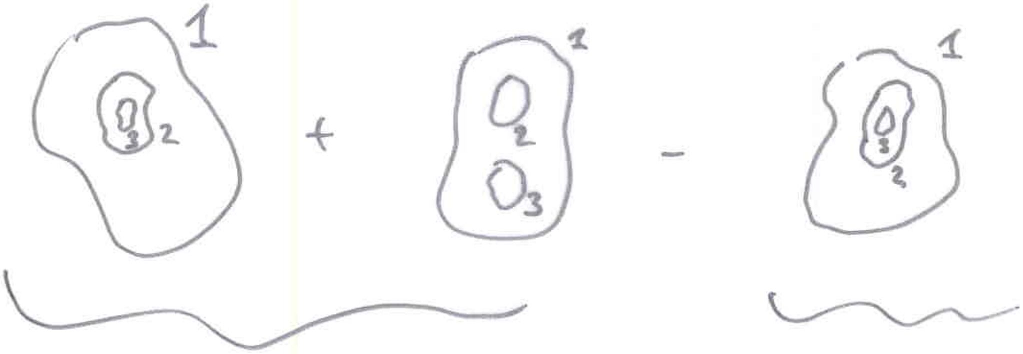
Where does this come from?

$$\Gamma_1 * \Gamma_2 = \text{"2 into 1"}$$

$$(\Gamma_1 * \Gamma_2) * \Gamma_3 - \Gamma_1 * (\Gamma_2 * \Gamma_3)$$

$$= \Gamma_1 * \{\Gamma_2 \cup \Gamma_3\}$$

$$= (\Gamma_1 * \Gamma_3) * \Gamma_2 - \Gamma_1 * (\Gamma_3 * \Gamma_2)$$

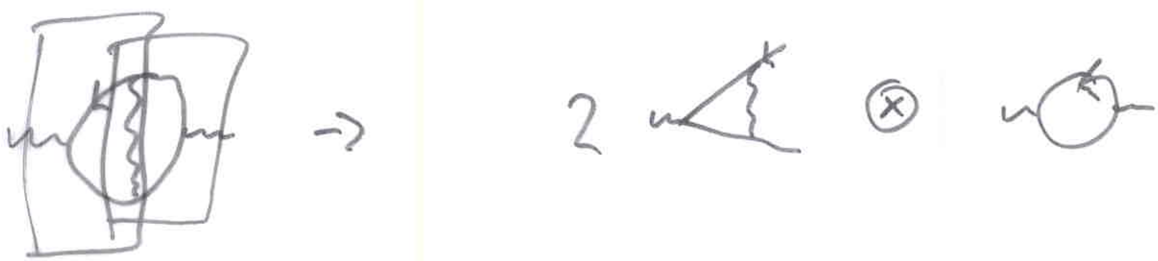


$(\Gamma_1 * \Gamma_2) * \Gamma_3$ $\qquad\qquad$ $\Gamma_1 * (\Gamma_2 * \Gamma_3)$

O.k., we can insert graphs,

so what ?

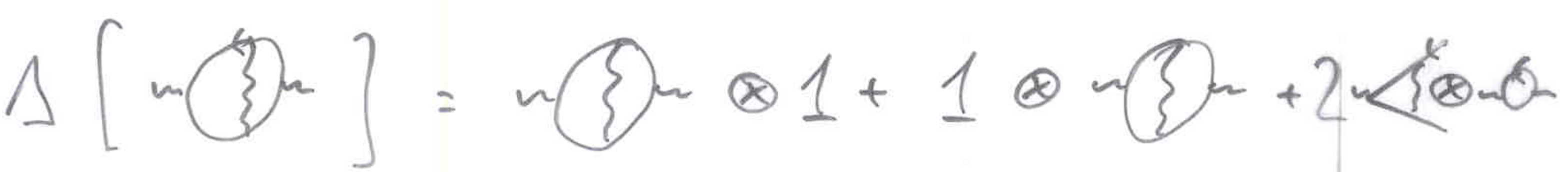
We can also decompose them!



→ the art of decomposition

products $a \times b \to a \cdot b$ coproduct: $X \to X' \otimes X''$

$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ "$\Delta[H]$" = "$H \otimes H$"



$m[\Gamma_i, \Gamma_j] = \Gamma_i \Gamma_j$

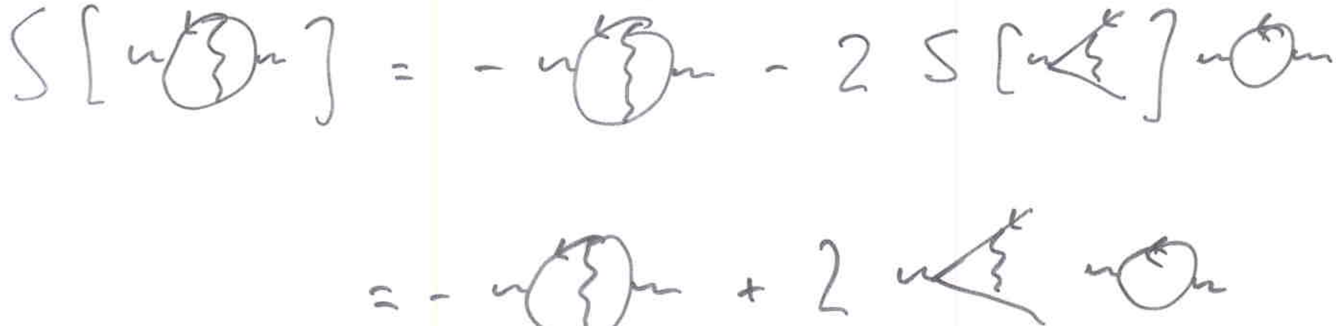
$\Delta[\Gamma] = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$

$1^*(\Gamma) = 0 \quad 1^*(1) = 1.$

... and there is a coinverse as well

$$S[e] = e$$

$$S[\Gamma] = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma)\, \Gamma/\gamma$$



→ bring in Feynman rules

$$\phi[\Gamma] = \text{"}\int\text{"} \qquad \phi : H \to V$$

$$\phi(\Gamma_1 \Gamma_2) = \phi(\Gamma_1)\,\phi(\Gamma_2)$$

We have a problem ☺ !

And can solve it!

let $R : V \to V$ be a Baxter map.

$$R[ab] + R[a]\,R[b] = R[a\,R[b]] + R[\,R[a]\,b\,].$$

$$S_R^{\phi} : H \to V$$

$$S_R^{\phi}(\Gamma) = -R\Big[\phi(\Gamma) + \sum_{\gamma \subset \Gamma} S_R^{\phi}(\gamma)\,\phi(\Gamma/\gamma)\Big]$$

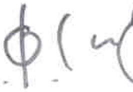
↑ twisted by R.

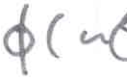
$$\phi \circ S(\Gamma) = -\Big[\phi(\Gamma) + \sum_{\gamma \subset \Gamma} \phi \circ S(\gamma)\,\phi(\Gamma/\gamma)\Big]$$

$\psi : H \to V$ $\qquad$ $\phi : H \to V$

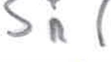
$\psi * \phi : H \to V$

$$\psi * \phi(\Gamma) = m_V \circ (\psi \otimes \phi) \circ \Delta[\Gamma].$$

$S_R^{\phi} * \phi$ () $=$ ϕ() $+$ S_R^{ϕ}()

$+\, 2\, S_R^{\phi}$() ϕ().

S_R^{ϕ}() $= -R[\phi$()$] - 2R[S_R^{\phi}$() ϕ()$]$

S_R^{ϕ}() $= -R[\phi[$  $]]$

finiteness a consequence of properties of R.

$$S_{id}^{\phi} \equiv \phi \circ S$$

$$S_{id}^{\phi} * \phi\,(\Gamma) = 0.$$

R faithful along divergences

$\rightarrow$ $S_{R}^{\phi} * \phi\,(\Gamma)$ free of divergences.

"principle of multiplicative subtraction"

- additive subtraction:

A has problem, B has same problem,

$A - B$ no problem.

- mult. subtr.: use group of characters

in general, add. subtr. special case.

Now, regulate, and subtract minimally.

$$S_R^\phi(\Gamma) = \sum_{k=-r_\Gamma}^{-1} c_k z^k \qquad \phi(\Gamma) = \sum_{k=-r_\Gamma}^{+\infty} b_k z^k$$

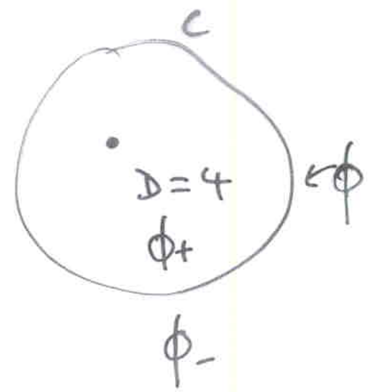
$$S_R^\phi * \phi(\Gamma) = \sum_{k=0}^{\infty} d_k z^k$$

- some general properties in complex Lie groups
- algebraic description of Diff
- universality of underlying Hopf algebra of rooted trees

→ understanding of the renormalization group, resolution of higher poles in residues

3)

swift to complex regularization



$\phi = \phi_-^{-1}\,\phi_+$ $\qquad$ $\phi_+ = \phi_-\,\phi$

$= S^{*}_{R=\Omega S}$ Feynman Rules

↑ minimal subtr.

Birkhoff decomposition

↳ solves Riemann Hilbert problem

[$\exists$ Diff. equation whose monodromy around $D=4$ is prescribed]

ϕ_- remains invariant under scale variations

↳ $g_{eff} = Z_1 Z_2^{-3/2}\, g$

defines Hopf algebra homomorphism

Diff of physical parameters $\Longleftarrow$ Hopf alg. of Feynman graphs

$Z_1 = 1 + (\;\;) + \ldots$ $\qquad$ $\Gamma \to e^{-t\,\mathrm{grad}[\Gamma]}\,\Gamma \equiv \Theta(\Gamma)$

$[\phi\circ\Theta]^{-1} * \phi$ is finite

$Z_2 = 1 + (\;\;) + \ldots$ $\qquad$ infinitesimal, pulled back $\beta_{g_{eff}}$

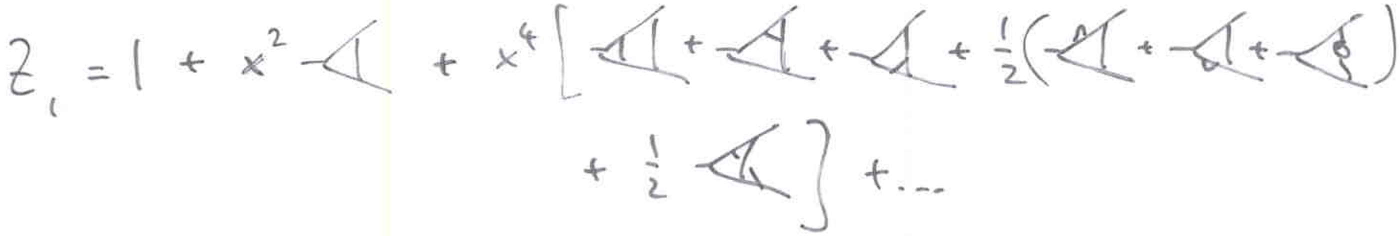
Only works for combination of Z-factors

Let

$$Z_1 = 1 + \sum z_{1,2k}\, x^{2k}$$

$$Z_2 = 1 + \sum z_{2,2k}\, x^{2k}$$

$$g(x) = x Z_1 Z_2^{-3/2}$$ regarded as a diffeom.

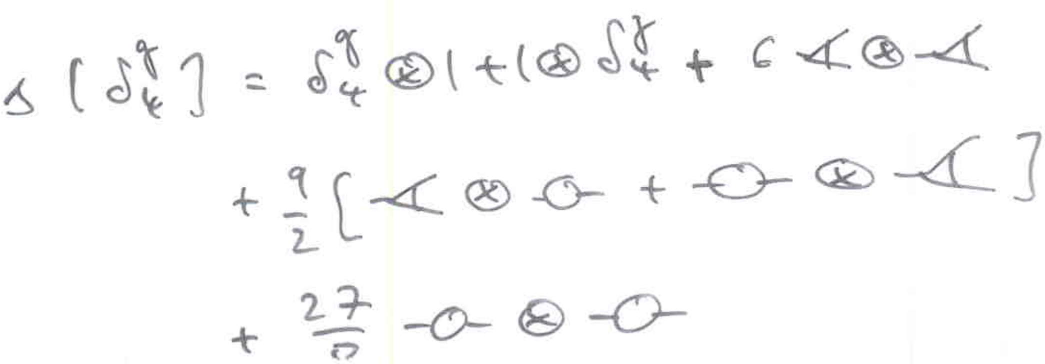


$$\log\left(\frac{\partial}{\partial x}\, x\, Z_1 Z_2^{-5/2}\right)$$

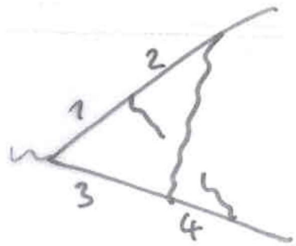
$$\delta_2^g = 3 z_{1,2} - \frac{9}{2} z_{2,2}$$

$$\delta_4^g = 5\left[z_{1,4} - \frac{3}{2} z_{2,4}\right] - \frac{7}{2} z_{1,2}^2 + 6 z_{1,2} z_{2,2} - \frac{3}{4} z_{2,2}^2$$

$$\Delta_{GR}[\delta_4^g] = \delta_4^g \otimes 1 + 1 \otimes \delta_4^g + \frac{2}{3}\delta_2^g \otimes \delta_2^g$$

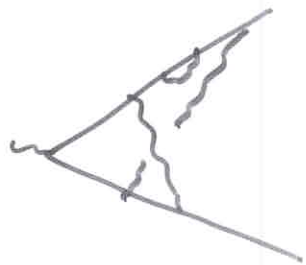


Feynman graphs as insertion operads



$\circ_2$

$=$

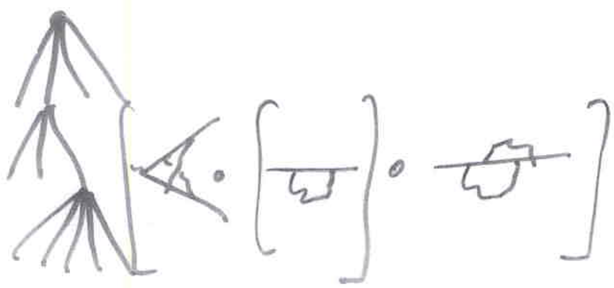


Γ provides places

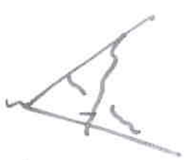
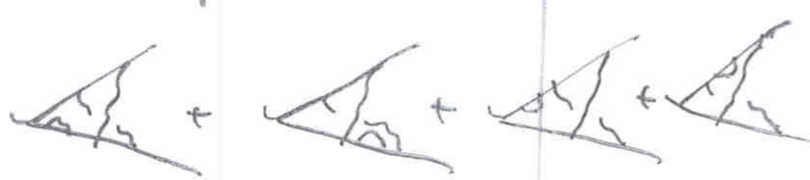
$\Leftrightarrow$ $\Gamma = \Gamma[\ldots\ldots]$

unique composition

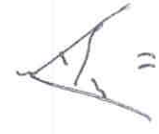
$\Leftrightarrow$



associativities, equivariant actions of the permutation group $\Leftrightarrow$ operad.

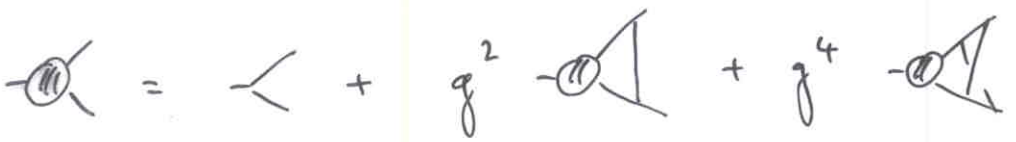
$\sum_{\text{places } i}$  $\circ_i$  $=$ 

$=:$  $*$ 

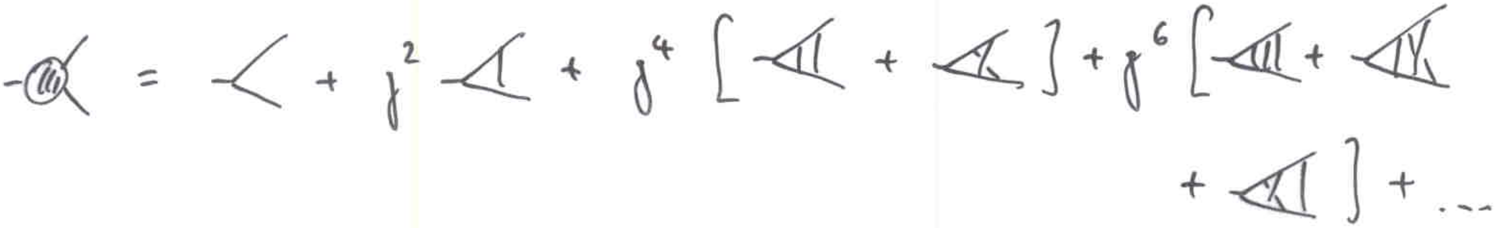
 $*$  $=$ 

Dyson-Schwinger Equations

& unique factorization



(linearized form)

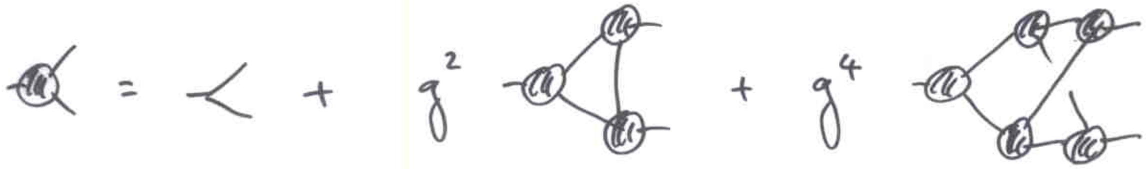


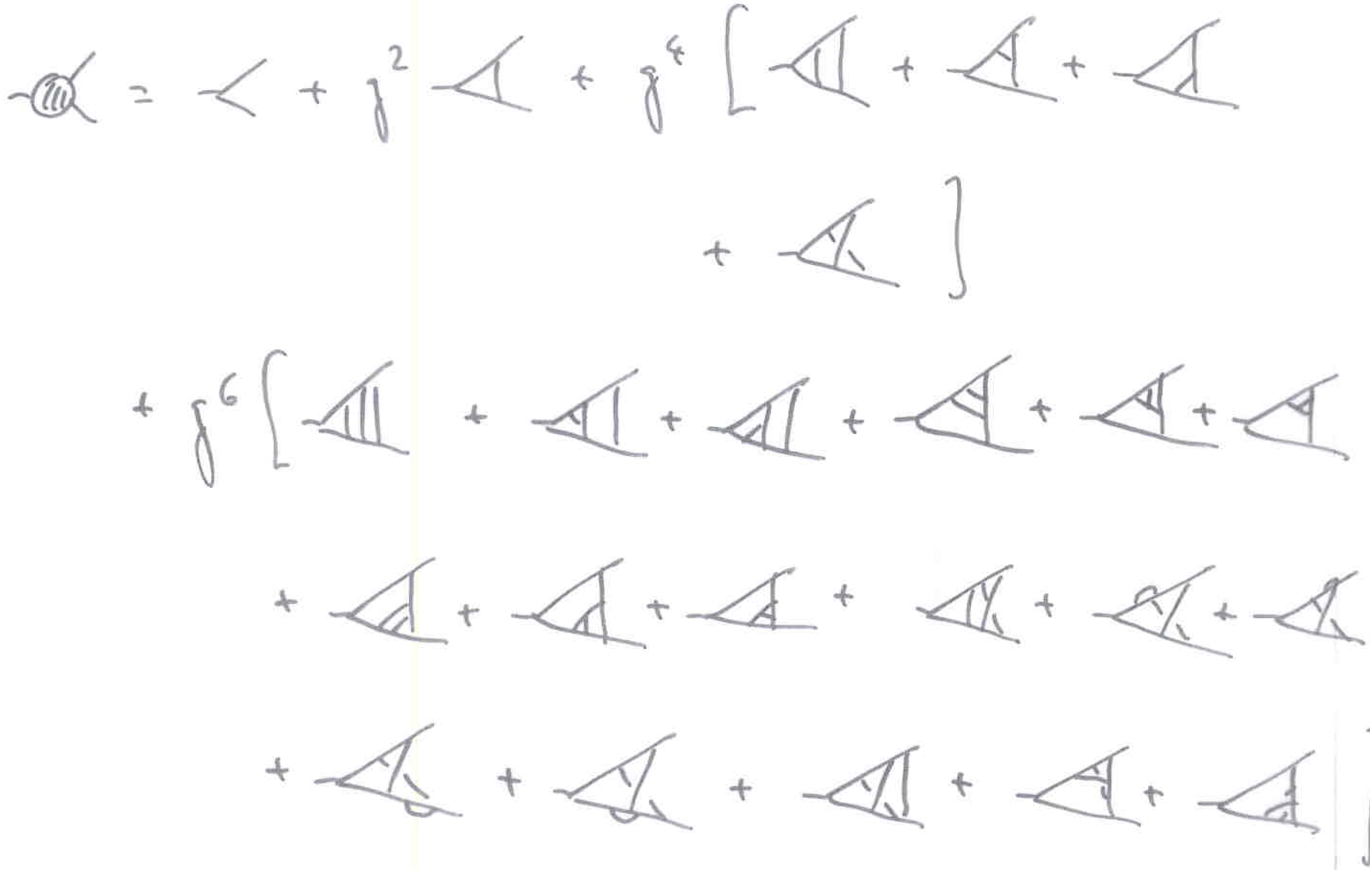
"p" prime graphs

$$\to \; 1 + g^2 a + g^4 [b + aa] + g^6 [ab + ba + aaa] + g^8 [aaaa + \underbrace{aab + aba + baa}_{aa \,ш\, b \,\hat{=}\, \frac{1}{2} a \,ш\, a \,ш\, b} + bb] + \ldots$$

 $= \prod_p^{ш} \frac{1}{1-p}$ ← Euler product

the non-linear case





$+ \dots$

a sequence of prime graphs $I = (\overbrace{\gamma_1, \gamma_2, \dots}^{k \text{ elements}})$

compatible with Γ, $I \sim \Gamma$

$\Leftrightarrow \langle \Delta^{k-1}(\Gamma), \gamma_k \otimes \dots \otimes \gamma_1 \rangle = 1.$

$$\sum_{\Gamma \sim I} \frac{1}{n_\Gamma} \Gamma = \underline{\Gamma} \qquad \underline{\Gamma_1} \cdot \underline{\Gamma_2} = \underline{\Gamma}(I_1 \cup I_2)$$

n_Γ: # of seq comp. with Γ

← generalized shuffle product

 $= \prod_p^{\bullet} \frac{1}{1-p}$

$\phi($  $) \stackrel{?}{=} \prod_p \frac{1}{1-\phi(p)}$

(physicist's dream)

. it's true for leading coefficient of divergence

→ construct a "factorization scheme"!

→ steps involved: compensate all n-point subdivs by functions which have unique scaling (2-pt).

→ construction of automorphisms of H determine the relevant anom. dim.

→ regulate by them

$$\frac{d \log Z}{d \log \mu^2} \iff S * Y$$

grading operator of H

antipode of H

$$q^2 \Sigma(q^2) = \frac{a}{\pi^2} \int \frac{d^4 l}{l^2(1-\Sigma(l^2))} \frac{q \cdot l}{(q+l)^2}$$

generated from $X = a B_+\left(\frac{1}{1-X}\right)$

anom. dim.

$$\hookrightarrow \quad \left.\frac{d \log(1-\Sigma)}{d \log q^2}\right|_{\log q^2 = 0} = \gamma\left(a/(1-\Sigma)^2\right)$$

$$\underset{\in H}{X} = a B_+\left(\frac{1}{1-X}\right) = a \underset{\in H}{X_1} + a^2 \underset{\in H}{X_2} + a^3 \underset{\in H}{X_3} + \dots$$

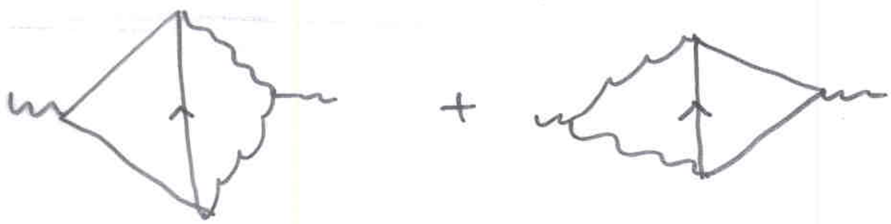
"befooting"

$$F_{X_m}(X_k) = [2(k-m)-1] X_{k-m}$$

$$\sigma^n_m = \lim_{\varepsilon \to 0} \varepsilon^n \left[\phi \circ (S * Y^n)\right](X_m)$$

$\hat{=}$ coeff. of $\left(\log \frac{q^2}{\mu^2}\right)^n$ in Σ

$$\sigma^n_m = \sum_{i=1}^{m-1} 2[(m-i)-1]\, \sigma^1_i\, \sigma^{n-1}_{m-i}$$



two factorizations!

Can we count?

$\mathbb{Q}(\sqrt{-5})$

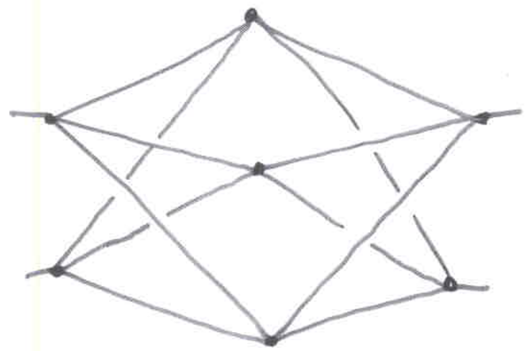
$$6$$

$$2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

this leads to considerations of ideals in gauge theories, much more then to be learned. [→ group theory, Galois theory ...]

Any idea?

$$\zeta(7) = \sum \frac{1}{n^7}$$



$\sim \zeta(5,3)$

as ϕ^4 in 4 D, this is an invariant contribution to the β-function.

"a residue"

most non-planar graphs – most interesting transcendentals

[at 7 loops, even planar graphs are "interesting", the non-planar ones "fascinating"]

Conclusions

- Lie and Hopf algebra structures in perturbative expansions
- universal combinatorial structures
- identify a QFT by its number-content
- develop a representation theory
- how do "existent" QFTs fare?
- much to be learned soon