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On representations of contact superconformal algebras.

distribution : $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$

Examples $\mathfrak{g} = W(N)$, $K(N)$, $S(N)$
$\cup$
Witt

Let $\Lambda(N)$ – the Grassmann algebra in $\theta_1, \theta_2, \dots, \theta_N$

$$\Lambda_t(N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$$

$$W(N) := \operatorname{Der} \Lambda_t(N) = \left\{ D = f\frac{\partial}{\partial t} + \sum_{i=1}^{N} f_i \frac{\partial}{\partial \theta_i} \;\middle|\; \begin{matrix} f \\ f_i \end{matrix} \in \Lambda_t(N) \right\}$$

$$\Omega = dt - \sum_{i=1}^{N} \theta_i d\theta_i \quad \text{– diff 1-form}$$

$$K(N) := \{ D \in W(N) \mid D\Omega = g\Omega \text{ for some } g \in \Lambda_t(N) \}$$

gen 2^N fields

$$f \in \Lambda_t(N) \longleftrightarrow D_f$$

s.t. $[D_f, D_g] = D_{\{f,g\}}$.

Let $\Delta(f) = 2f - \sum_{i=1}^{N} \theta_i \frac{\partial f}{\partial \theta_i}$

Then $D_f = \Delta(f)\frac{\partial}{\partial t} + \frac{\partial f}{\partial t} \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i} + (-1)^{p(f)N} \sum_{i=1}^{N} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i}$

$$\{f, g\} = \Delta(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\Delta(g) + (-1)^{p(f)} \sum_{i=1}^{N} \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i}$$

$K(N)$ is simple if $N \neq 4$

$\hat{K}(1)$ the N-S algebra

$\hat{K}(2)$ the $N=2$ SCA

If $N=4$, then $K'(4) = [K(4), K(4)]$ is simple.

$\hat{K}'(4)$ – the "big $N=4$ SCA"

$N>4$ → no central extensions.

$K'(N)$ acts in a natural way on the space of "densities"

$$V_{\lambda,\mu} = \{t^\mu g \Omega^\lambda\}, \quad \mu, \lambda \in \mathbb{C}, \quad g \in \Lambda_+(N)$$

$$D_f(t^\mu g \Omega^\lambda) = \left(D_f(t^\mu g) + (-1)^{p(f)p(g)} \cdot 2\lambda t^\mu g \frac{\partial f}{\partial t}\right)\Omega^\lambda$$

<u>V. Kac</u>.

$$P = \left\{ \sum_{-\infty}^{n} a_i(t) \xi^i \,\middle|\, a_i(t) \in \mathbb{C}[t, t^{-1}] \right\}$$ – the Poisson alg of pseudodiff symbols on S^1

$$\xi \leftrightarrow \frac{\partial}{\partial t}$$

$$\{A(t,\xi), B(t,\xi)\} = \frac{\partial A}{\partial \xi}\frac{\partial B}{\partial t} - \frac{\partial A}{\partial t}\frac{\partial B}{\partial \xi}$$

P_h, $h \in]0,1]$

$$A(t,\xi) \circ_h B(t,\xi) = \sum_{n \geq 0} \frac{h^n}{n!} \partial_\xi^n A(t,\xi)\, \partial_t^n B(t,\xi)$$

$$\xi^{-1} \circ_h a(t) = a(t)\xi^{-1} - h a'(t)\xi^{-2} + h^2 a''(t)\xi^{-3} + \cdots$$

$$[A, B]_h = \frac{1}{h}(A \circ_h B - B \circ_h A)$$

$$\lim_{h \to 0} [A, B]_h = \{A, B\}.$$

<u>Poisson superalgebra</u> $P(N) = P \otimes \Lambda(2N)$

$$\Lambda(2N) = \langle \theta_i, \ \bar{\theta}_i \text{ "=" } \frac{\partial}{\partial \theta_i} \rangle$$

$$\{A, B\}_{P.s.b} = \{A, B\}_{P.b.} - (-1)^{p(A)} \left(\sum_{i=1}^{N} \frac{\partial A}{\partial \theta_i} \frac{\partial B}{\partial \bar{\theta}_i} + \frac{\partial A}{\partial \bar{\theta}_i} \frac{\partial B}{\partial \theta_i} \right)$$

$$P_h(N) = P_h \otimes \Lambda_h(2N)$$

$$\theta_i \theta_j = -\theta_j \theta_i, \quad \bar{\theta}_i \bar{\theta}_j = -\bar{\theta}_j \bar{\theta}_i$$

$$\bar{\theta}_i \theta_j = h\,\delta_{ij} - \theta_j \bar{\theta}_i$$

$$A \circ_h B = \frac{1}{h}(A_1 \circ_h B_1) \otimes XY$$

$$A = A_1 \otimes X, \quad B = B_1 \otimes Y$$

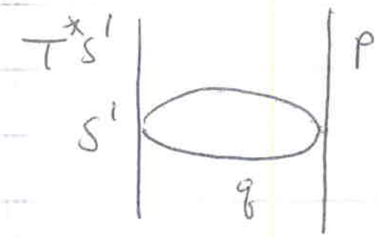
$$\lim_{h \to 0} [A, B]_h = \{A, B\}.$$

$$W(N) \hookrightarrow P(N)$$

$$W(N) \hookrightarrow P_h(N)$$

There is an embedding $K(2N) \hookrightarrow P(N)$, $N \geq 0$

<u>Remark</u>: $N = 0$ $\quad Vect(S^1) \hookrightarrow$ Poisson algebra of functions on the cylinder $\overset{\circ}{T^*S^1} = T^*S^1 \setminus S^1$



$$(q, p) = (t, \xi), \quad \omega = dp \wedge dq$$

$$A(q, p) \to H_A = \frac{\partial A}{\partial p}\frac{\partial}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial}{\partial p}$$ V

$$f(q)\frac{\partial}{\partial q} \longrightarrow H_{f(q)p}$$

$$[H_A, p\frac{\partial}{\partial p}] = 0.$$

$N \geq 1$, $P(N)$ $\qquad q, \; p, \; \theta_i, \; \overline{\theta_i}$ $\quad$ ($q = t$, $p = \xi$)

$\mathbb{Z}$-grading $\quad \deg \xi = \deg \overline{\theta_i} = 1 \quad (i = 1, \cdots, N)$.

$\qquad \deg t = \deg \theta_i = 0$

$$P(N) = \bigoplus_{j \in \mathbb{Z}} P_j(N) \qquad \{P_j(N), P_k(N)\} \subset P_{j+k-1}(N).$$

subalgebra. $P_1(N) \cong K(2N)$

$$P_1(N) = \left\{ A(q, p, \theta_i, \overline{\theta_i}) \,\middle|\, [H_A, p\frac{\partial}{\partial p} + \sum_{i=1}^{N} \overline{\theta_i}\frac{\partial}{\partial \overline{\theta_i}}] = 0 \right\}$$

Question: Is there embedding

$$K(2N) \hookrightarrow P_h(N) \,?$$

$N = 1$. $\quad K(2) \cong W(1) \hookrightarrow P_h(1)$

$N = 2$ $\quad \hat{K}'(4) \hookrightarrow P_h(2) \qquad \lim_{h \to 0} \hat{K}'(4) = K'(4) \hookrightarrow P(2)$

Center is h

√ $N \geq 3$ $\quad$ NO.

<u>Let $N=2$</u> 1-parameter family of superalgebras

$$S(2,\alpha) = \{ D \in W(2) \mid \mathrm{Div}(t^{\alpha} D) = 0 \} \qquad \alpha \in \mathbb{C}.$$

For $\alpha \in \mathbb{Z}$, $S'(2,\alpha)$ are simple and isom. to each other.

$$0 \to S'(2,\alpha) \to S(2,\alpha) \to \mathbb{C}\, t^{-\alpha}\xi\theta_1\theta_2 \to 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; -F_{-\alpha+1}$$

$$\hat{S}'(2,0) = N=4 \text{ SCA.}$$

<u>Prop 1</u> $\mathrm{Der}_{ext}\, S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ generate the

$$\widetilde{sl}(2) = \langle E_n, H_n, F_n \rangle \subset P(2).$$

$$\mathrm{Der}_{ext}\, S'(2,\alpha) = \langle E_{\alpha-1}, H_0, F_{-\alpha+1} \rangle$$

$$E_{\alpha-1} = t^{\alpha}\xi^{-1}\bar{\theta}_1\bar{\theta}_2$$

$$H_0 = -\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2$$

2) The family $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ together with $\widetilde{sl}(2)$ generate a Lie s. algebra isomorphic to $K'(4) \subset P(2)$

$$0 \to K'(4) \to K(4) \to \mathbb{C}\, t^{-1}\xi^{-1}\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2 \to 0.$$

<u>Prop 2</u>. The similar construction for $h \in]0,1]$ gives an embedding

$$\hat{K}'(4) \subset P_h(2) \qquad (\text{center is } h). \qquad (*)$$

<u>Remark</u> $K'(4)$ has 3 indep. central extensions (V. Kac, V. de Leur)

P_h has 2 indep central ext: $\log \xi$, $\log t$

Let $x = \xi, t$, $\hat{x} = t, \xi$

$$[\log x, A(t,\xi)] = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \partial_{\hat{x}}^{k} A(t,\xi) x^{-k}$$

$P_h(2)$ $\quad C_x(A \otimes X, B \otimes Y) =$ coeff of $t^{-1}\xi^{-1}\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$ in $([\log x, A] \circ_h B) \otimes XY$

The cocycle in $(*)$ is equivalent to the restriction of C_t-cocycle.

<u>Theorem</u> There exists a 1-parameter family of spinor-like repns of $K'(4)$ realized on 4 fields.

$\hat{K}'(4) \subset P_h(2)$ is spanned by $W(2)$ and 4 fields.

$$F_n^0 = (\xi^{-1} \circ_h t^n)\bar{\theta}_1\bar{\theta}_2$$

$$F_n^i = (\xi^{-1} \circ_h t^n)\bar{\theta}_1\bar{\theta}_2\theta_i \quad , \quad i = 1, 2$$

$$F_n^3 = (\xi^{-1} \circ_h t^n)\bar{\theta}_1\bar{\theta}_2\theta_1\theta_2 + \frac{h}{n+1} t^{n+1} \qquad n \neq -1$$

and the center h.

<u>Fix $h = 1$</u>

ξ^{-1} is an anti-deriv. on the space

$$t^{\mu}\mathbb{C}[t, t^{-1}] , \quad \mu \in \mathbb{R} \setminus \mathbb{Z}$$

$$\xi^{-1} \circ f = \sum_{n=0}^{\infty} (-1)^n (\xi^n f) \xi^{-n-1}$$ – integration by parts.

Let $V^\mu = t^\mu \Lambda_t(2)$

Basis: $v_m^0 = \frac{1}{m+\mu} t^{m+\mu}$

$v_m^i = t^{m+\mu}\theta_i \quad , \quad i=1,2$

$v_m^3 = t^{m+\mu}\theta_1\theta_2$

$W(2)$ acts on V^μ by derivatives

$F_n^0(v_m^3) = -v_{m+n+1}^0$

$F_n^1(v_m^2) = -v_{m+n+1}^0$

$F_n^2(v_m^1) = v_{m+n+1}^0$

$F_n^3(v_m^i) = \frac{1}{n+1} v_{m+n+1}^i \qquad i=0,1,2,3 \ , \ n \neq -1$

The center is 1 and it acts by the identity operator

$CK_6 \hookrightarrow K(6) \hookrightarrow P(3)$

spanned by 32 fields

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$CK_6 \overset{?}{\hookrightarrow} P_h(3). \qquad t^\mu \Lambda_t(3)$

Is there a repres. realized on 8 fields?