M. Verbitsky

Hodge theory and supersymmetry
on Kähler, hyperkähler, and hyperkähler torsion mflds.

1. Supersymmetry of Kähler and hyperkähler manifolds
2. HKT manifolds.
3. Use supersymmetry to study HKT-manifold

M – Riemannian mfld

d – de Rham

d^* – ad, ort operator

$\{d, d^*\} = \Delta$ – Laplacian.

$I: TM \to TM \quad I^2 = -1$ Cpx str.

g – Riemannian str. orthogonal.

$\omega = g(\cdot, I\cdot)$. 2-form

Kähler: $d\omega = 0$.

M Kähler $\quad L: \eta \to \eta \wedge \omega \quad \Lambda^p(M) \to \Lambda^{p+2}(M)$

Λ – hermitian adjoint.

$[L, \Lambda] = H \qquad H(\eta) = (n-p)\eta$

$\eta \in \Lambda^p(M), \quad n = \dim_{\mathbb{C}} M$.

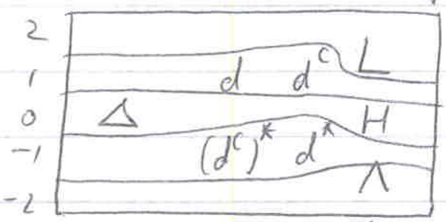
$[H, d] = d$

$[L, d] = 0$

d – belongs to weight 1 representation.

$d^c := I \circ d \circ I^{-1}$ $\qquad [\Lambda, d^c] = d^*, \quad [\Lambda, d] = (d^c)^*$

2 $\quad$ 2-dim representations of $SL(2)$



8-dimensional

$d^2 = 0$

$(d^c)^2 = 0$

$\{d, d^c\} = 0$. (integrability condition for complex structure)

$$0 = [\Lambda, \{d, d\}] = 2\{(d^c)^*, d\} = 0.$$

Similarly from $\{d, d^c\} = 0 \Rightarrow \{d, d^*\} = \{d^c, d^{c*}\}$.

Hyperkähler

M – Riemannian.

$\mathbb{H}$ action on TM

(I, J, K).

$\omega_I = g(\cdot, I\cdot)$

$\omega_J = g(\cdot, J\cdot)$

$\omega_K = g(\cdot, K\cdot)$

hyperkähler: $d\omega_I = d\omega_J = d\omega_K = 0$

$Spin(4,1) = SU(\mathbb{H}, 1, 1)$, as Lie algebra this is $so(4,1)$.

$$d. \quad d_I \quad d_J \quad d_K$$
$$\Delta \qquad d_I = d^c$$
$$d^* \quad d_I^* \quad d_J^* \quad d_K^*$$

$$\boxed{\begin{array}{lll} L_I, & L_J, & L_K \\ H & & su(2) = \mathrm{Lie}(\mathbb{H}). \\ \Lambda_I, & \Lambda_J, & \Lambda_K \end{array}}$$

$\dim(11|8)$.

Wt system of this algebra.
$\Leftrightarrow$ Hodge numbers

Hodge diamond for hyperkähler manifold of $\dim_{\mathbb{R}} 8$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & a & & 1 & & \\ & 0 & & b & & b & & 0 & \\ 1 & & a & & 0 & & a & & 1 \\ & 0 & & b & & b & & 0 & \\ & & 1 & & a & & 1 & & \\ & & & & 1. & & & & \end{array}$$

$\Omega = \omega_J + \sqrt{-1}\,\omega_K$

This is a $(2,0)$-form on (M, I).

$L_\Omega \eta = \eta \wedge \Omega$

$\Lambda_\Omega = \mathrm{adj}$ on T

$H_\Omega : [\Lambda_\Omega, \Lambda_\Omega]$ $\quad$ ((p,q) form multiplied by $n-p$, $n = \dim_{\mathbb{H}} M$.

∂: Dolbeault differential

$$\partial = \frac{d + \sqrt{-1}\, d_I}{2}$$

$$(\bar{\partial})^J := \partial_J : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$$

$$\boxed{\begin{array}{ccc} & \partial \quad \partial_J & L_\Omega \\ \Delta & & H \\ & \partial^* \quad \partial_J & \Lambda_\Omega \end{array}}$$ on $\Lambda^{p,0}(M)$.

$(p,0)$ hyperkahler behaves like p forms in kahler.

HKT manifold

I, J, K — quarternionic action on TM

$\mathbb{H}$ action. integrable.

hypercomplex.

hypercomplex Hermitian is Riemannian.

Chern-Gauduchon connection.

M. $\nabla : TM \otimes TM \longrightarrow TM$ $\quad \nabla_X Y$.

$$\nabla_X Y - \nabla_Y X + [X, Y] = \underbrace{\text{torsion}}_{T.} \text{ of } \nabla$$

$$T \in \Lambda^2 M \otimes TM.$$

$$T^{new} \in \Lambda^2 M \otimes \Lambda^1 M$$

antisymmetric torsion:

$$T \in \Lambda^3 M \subset \Lambda^2 \otimes \Lambda^1$$

Gauduchon:

Let M be complex mfld
Hermitian.

Then $\exists!$ $\underbrace{\text{torsion}}_{\text{antisym}}$ connection preserving complex and Hermitian str.

∇_I – Gauduchon connection on (M, I).

∇_J, ∇_K "

HKT: $\boxed{\nabla_I = \nabla_J = \nabla_K}$

Example. $SU(2) \times U(1) = \mathbb{H}^* / \mathbb{Z}$ $\quad q \to q^\lambda$, $\lambda \in \mathbb{R}^{>1}$

This is HKT mfld.

$SU(3)$. $\supset SU(2) \times U(1)$.

Lie alg $\begin{pmatrix} su(2) & \cdot \\ \cdot & u(1) \end{pmatrix}$ has quaternionic str

Joyce: Every compact Lie group G becomes HKT if multiplied by $(S^1)^k$

Ω as before. $\partial\Omega \in \Lambda^{3,0}(M, I)$.

HKT: $\boxed{\partial\Omega = 0}$

Canonical class K nontrivial.

$$\Delta \quad \begin{array}{c} \partial, \partial_J \\ \\ \delta, \delta_J \end{array} \quad \begin{array}{c} L_\Omega \\ H \\ \Lambda_\Omega \end{array}$$

$\delta \neq \partial^*$

$\delta^* - \partial = \theta$

$\theta = \partial_J^* \Omega$

= connection term in K.

Another superalg action

$$\Delta \quad \begin{array}{c} \delta^* \ \delta_J^* \\ \\ \partial^* \ \partial_J^* \end{array} \quad \begin{array}{c} L \\ H \\ \Lambda. \end{array}$$

Take average of the two $\ ^n\partial = \frac{\partial + \delta^*}{2}$, etc.

$$^n\Delta \quad \begin{array}{cc} ^n\partial & ^n\partial_J \\ ^n\partial^* & ^n\partial_J^* \end{array} \quad \begin{array}{c} L \\ H \\ \Lambda \end{array}$$

$^n\Delta$ — Hermitian adjoint. → computes cohomology of $^n\partial$.

$\ker {}^n\Delta = H^*(M, \sqrt{K})$.

$\Uparrow$ so has action of above algebra

= Harmonic spinors.