FINSLER GEOMETRY IN COMPLEX ANALYSIS

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AIMS OF THE TALK:

Illustrate:

– how Finsler metrics appear naturally in Complex Analysis as tools in function theory and in classification problems;

– some results to show how Finsler geometry may be of use (when not otherwise stated joint work with **Marco Abate**);

– an important special example: Teichmüller spaces emphasizing the role of the curvature and trying to show how it is important to develop Finsler geometry under weak assumptions of regularity

Uniformization of Riemann Surfaces:

M Riemann surface with $\Pi_1(M) = 0$. Then

	Function theory	*Potential theory*	*Differential geometry*
$M \simeq \Delta$	M carries noncostant bound holo functions	M has Green function	M carries compl metric of constant neg curvature
$M \simeq \mathbb{C}$	M non cmpct, with no noncostant bound holo functions	M has parabolic potential	M carries complete flat metric
$M \simeq \mathbb{P}^1$	M is compact	$M - \{pt\}$ has parabolic potential	M carries metric of constant pos curvature

M arbitray Riemann surface. Then

either	$M = \Delta/\Gamma$	*Hyperbolic*
or	$M = \mathbb{C}/\Gamma = \mathbb{C}, \mathbb{C}^*$, Torus	*Parabolic*
or	$M = \mathbb{P}^1$	*Elliptic*

In higher dimension?

Much more to do:

- simply connected, complete Khäler manifold of constant curvature are $\mathbb{C}^n, \mathbb{B}^n, \mathbb{P}^n$ **but**

- not known if a simply connected, complete hermitian manifold of constant negative curvature is Khäler

- almost no simply connected bounded domain in $\mathbb{C}^n$ is biholomorphic to $\mathbb{B}^n$

- no easy way to classify manifolds which carry bounded holomorphic functions

- the potential theory of the highly non linear *homogeneous Monge-Ampère* equation

$$(\partial\overline{\partial}u)^n = 0$$

replaces the potential theory of Laplace equation but the results are much weaker

- classification is open question even for very special classes of manifolds: algebraic, projective, domains with many symmetries.....

Intrinsic metrics, a useful tool:

M complex manifold $p \in M$, $v \in T_p^{1,0}M$

Kobayashi (pseudo)metric:

$$\kappa_M(p;v) = \inf\{|\xi| \mid \exists \varphi \in Hol(\Delta, M) : d\varphi_0(\xi) = v\}$$

- In general it is an uppersemicontinuous Finsler (pseudo)metric

- *Kobayashi (pseudo)distance:* defined by the integrated lenght of κ_M

Carathéodory (pseudo)metric:

$$\gamma_M(p;v) = \sup\{|df_p(v)| \mid \exists f \in Hol(M, \Delta) : f(p) = 0\}$$

- In general it is a continuous Finsler (pseudo)metric

Facts:

1. $\kappa_\Delta = \gamma_\Delta =$ Poincaré metric.
2. $\gamma_M \leq \kappa_M$ for all M
3. Biholomorphisms are isometries for κ_M and γ_M ("invariance")
4. M is called (complete) hyperbolic if the Kobayashi pseudodistance is a (complete) distance
5. γ_M, κ_M hermitian "essentially" only for the ball (and its quotients).

1. Notations

- M complex manifold, $p \in M$,
- $T^{1,0}M$ its holomorphic tangent bundle
- $\tilde{M}$ = complement in $T^{1,0}M$ of the Zero section.

coördinates $(z^1, \ldots, z^m)$ for M

$$\Downarrow$$

coordinates $(z^1, \ldots, z^m, v^1, \ldots, v^m)$ on $T^{1,0}M$

$$+$$

local frame $\{\partial_1 = \frac{\partial}{\partial z^1}, \ldots, \partial_n, \dot{\partial}_1 = \frac{\partial}{\partial v^1}, \ldots, \dot{\partial}_n\}$ for $T^{1,0}\tilde{M}$

Derivatives for ϕ C^∞ on an open set of $T^{1,0}M$:

$$\phi_{\alpha\bar{\beta}} = \frac{\partial^2 \phi}{\partial v^\alpha \partial \bar{v}^\beta}, \qquad \phi_{;\mu\nu} = \frac{\partial^2 \phi}{\partial z^\mu \partial z^\nu}, \qquad \phi_{\alpha;\bar{\nu}} = \frac{\partial^2 \phi}{\partial \bar{z}^\nu \partial v^\alpha}$$

2. What is a complex Finsler Metric?

– $F\colon T^{1,0}M \to \mathbb{R}^+$ **is a** *complex Finsler metric* **if**

F is an upper semicontinuous function,

$$F(p;v) > 0 \;\forall\, (p;v) \in \tilde{M},$$

$$F(p;\zeta v) = |\zeta| F(p;v) \qquad \forall\, (p;v) \in T^{1,0}M, \forall \zeta \in \mathbb{C}$$

– F is a *smooth* if

$$G = F^2 \in C^\infty(\tilde{M})$$

– F is a *smooth strongly pseudoconvex* if

$$\big(G_{\alpha\bar{\beta}}(p;v)\big) > 0 \;\forall\, (p;v) \in \tilde{M}$$

$$\Updownarrow$$

all indicatrices $I_F(p) = \{v \in T^{1,0}_p M \mid F(v) < 1\}$
are strongly pseudoconvex

Remark: $G = F^2$ hermitian iff $G \in C^\infty\left(T^{1,0}M\right)$

[redacted] *Complex Minkowski on $\mathbb{C}^n$*

$\tilde{g}: \mathbb{P}_{n-1} \to \mathbb{R}_+$ of class C^∞ with lifting $g: \mathbb{C}^n \to \mathbb{R}_+$ s.t.

$\tau(z) = g(z)||z||^2$ is strictly plurisuharmonic on $\mathbb{C}^n \setminus \{0\}$

$$\mu(z, v) = \sqrt{\tau(v)}$$

defines a smooth strongly pseudocvx complex Finsler metric

$$\mu: \mathbb{C}^n \times \mathbb{C}^n \simeq T^{1,0} \to \mathbb{R}_+$$

with vanishing holomorphic curvature and such that τ (= distance squared from 0) satisfies Monge-Ampère:

$$(\partial\overline{\partial} \log \tau)^n = 0.$$

[redacted] *Smooth Kobayashi metric* (Lempert '81)

$D \subset \mathbb{C}^n$ smooth bounded strictly convex domain

κ_D is a smooth strongly **convex** complex Finsler metric with constant negative holomorphic curvature, such that the distance δ_p from $p \in D$ satisfies Monge-Ampère:

$$(\partial\overline{\partial} \log \tanh \delta_p)^n = 0$$

(Pang '95) Also the Kobayashi metric of strongly pseudoconvex complete circular domains is smooth in a neighborhood of 0

3. Complex Finsler Gymnastics

–Vertical Bundle: $\mathcal{V} \to \tilde{M}$ **defined by:**

$$
\begin{array}{ccccc}
\mathcal{V} = Ker\, d\pi & \longrightarrow & T^{1,0}\tilde{M} & \xrightarrow{d\pi} & T^{1,0}M \\
 & \searrow & \downarrow & & \downarrow \\
 & & \tilde{M} & \xrightarrow{\pi} & M \\
 & & \downarrow & & \downarrow \\
 & & T^{1,0}M & \xrightarrow{\pi} & M.
\end{array}
$$

Local frame for $\mathcal{V}$: $\{\dot{\partial}_1, \ldots, \dot{\partial}_n\}$

radial vertical field: $\iota\colon \tilde{M} \to \mathcal{V}$ defined by:

$$\iota\left(v^\alpha \frac{\partial}{\partial z^\alpha}\right) = v^\alpha \dot{\partial}_\alpha$$

Hermitian metric on $\mathcal{V}$ defined by F:

$$\langle W, Z \rangle = G_{\alpha\bar{\beta}}(p; v) W^\alpha \bar{Z}^\beta$$

for $(p; v) \in \tilde{M}$ e $W, Z \in \mathcal{V}_{(p;v)}$

Main property: $\iota\colon \tilde{M} \to \mathcal{V}$ is isometry:

$$G(p; v) = G_{\alpha\bar{\beta}}(p; v) v^\alpha \bar{v}^\beta = \langle \iota(v), \iota(v) \rangle$$

It is defined

$$D: \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes \mathcal{V})$$

Chern connection, with covariant derivative ∇.

The *horizontal bundle* $\mathcal{H} \to \tilde{M}$ is defined by

$$\mathcal{H} = Ker X \mapsto \nabla_X \iota =$$

"subbundle of vectors respect to which ι is parallel"

Then $\qquad T^{1,0}\tilde{M} = \mathcal{V} \oplus \mathcal{H}$

Computing Chern connection get frames $\{\delta_1, \ldots, \delta_n\}$ for $\mathcal{H}$:

$$\delta_\mu = \partial_\mu - \Gamma^\alpha_\mu \dot{\partial}_\alpha = \partial_\mu - (G^{\bar{\tau}\alpha} G_{\bar{\tau};\mu})\dot{\partial}_\alpha$$

The *horizontal map* $\Theta: \mathcal{V} \ni \dot{\partial}_i \mapsto \delta_i \in \mathcal{H}$ defines:

– *horizontal radial vector field* $\chi = \Theta \circ \iota: \tilde{V} \to \mathcal{H}$

– metric on $T^{1,0}\tilde{M}$ prescribing $\mathcal{V} \perp \mathcal{H}$ and

$$\langle H, K \rangle = \langle \Theta^{-1}(H), \Theta^{-1}(K) \rangle \qquad \forall H, K \in \mathcal{H}_v$$

Basic tool of Complex Finsler Geometry:

The Chern connection of this metric called *Chern-Finsler connection* of F (first discovered using local considerations by Rund (1967)

General Philosophy: χ translates problems on $T^{1,0}M$ in questions on $\mathcal{H}$: deal with them using the Chern-Finsler connection

4. Curvature

Following Kobayashi ('75):

The usual procedure in hermitian geometry associates to the Chern-Finsler connection a *curvature operator*

$$\Omega \in \mathcal{X}(\bigwedge^2(T^*_{\mathbb{C}}\tilde{M}) \otimes \bigwedge^{1,0}\tilde{M} \otimes T^{1,0}\tilde{M})$$

with local expression

$$\Omega = \Omega^\alpha_\beta \otimes [dz^\beta \otimes \delta_\alpha + \psi^\beta \otimes \dot{\partial}_\alpha]$$

where $\Omega^\alpha_\beta = d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma$

ω^α_β: connection forms

$\{dz^\alpha, \psi^\beta\}$: frame dual to $\{\delta_\alpha, \dot{\partial}_\beta\}$

Definition. The <u>*holomorphic curvature*</u> of F along $v \in \tilde{M}$ is

$$K_F(v) = \frac{2}{[G(v)]^2}\langle\Omega\,(\chi(v),\bar{\chi}(v))\,\chi(v),\chi(v)\rangle$$

$$=_{(locally)} -\frac{2}{[G(v)]^2}G_\alpha(v)\,\left[G^{\alpha\bar{\nu}}(v)G_{\bar{\nu};\mu}(v)\right]_{\beta}v^\mu\bar{v}^\beta$$

Theorem 4.1: *For all $p \in M$ and $v \in \tilde{M}_p$*

$$K_F(v) = sup\{K(\phi^* G)(0)\}$$

where

$\phi \in Hol(\Delta, M)$ *with* $\phi(0) = p$ *and* $\phi'(0) = \lambda v$, *some* $\lambda \in \mathbb{C}^*$

$K(\phi^* G)(0)$: *Gauss curvature in* 0 *of* $\phi^* G$ *on the unit disk* Δ.

due to

....H.Wu '73 for hermitian metrics

....H Royden '84, Abate-Patrizio '94

This expression for the holomorphic curvature makes sense for uppersemicontinuous metrics using generalized Laplacian

$$\Delta u(\zeta) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(\zeta + re^{i\theta})\, d\theta - u(\zeta) \right\}$$

idea sistematically used by Heins in the 60's to study Gaussian curvature of pseudohermitian metrics instead of using supporting metrics

5. Non regular metrics

Little is known. For instance (Abate-Patrizio '93-'96):

Notation: μ_a complete metric on Δ constant curvature $= -4a$, for some $a > 0$.

Theorem 5.1: (Ahlfors' Lemma) *F complex Finsler metric on a complex manifold M with with holo curv $\leq -4a$, for some $a > 0$. Then*

$$\varphi^* F \leq \mu_a$$

for all holomorphic maps $\varphi\colon \Delta \to M$.

Proof: Maximum [redacted] principle built in the curvature!

Corollary 5.2: *M complex manifold with a (complete) continuous complex Finsler metric F with holo curv $\leq -4a$, for some $a > 0$. Then M is (complete) hyperbolic.*

Proof: May assume $K_F \leq -4$. d distance induced by F on M, ω the Poincaré distance on Δ. Then Ahlfors' Lemma yields

$$d\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \leq \omega(\zeta_1, \zeta_2),$$

for all ζ_1, $\zeta_2 \in \Delta$ and holomorphic maps $\varphi\colon \Delta \to M$. $\Rightarrow$ the Kobayashi distance k_M of M is bounded below by d, and the assertion follows.

Facts (B. Wong '77, Suzuki '83):

- Holomorphic curvature Kobayashi metric ≥ -4
- Holomorphic curvature Carathéodory metric ≤ -4

Naive question: If on M HOLCURV(Kobayashi) $= -4$, do Kobayashi and Carathéodory metric agree?

Lempert '83: True for convex domains in $\mathbb{C}^n$.

In general little is known:

Theorem 5.3: *F complex Finsler metric on a complex manifold M. Then*

$$F = \kappa_M \Leftrightarrow$$

(1) HOLCURV(F) $= \leq -4$ and
(2) for all $(p; v) \in T^{1,0}M$ $\exists \varphi \in Hol(\Delta, M)$ s.t.

$$\varphi(0) = p \text{ and } \lambda\varphi'(0) = v \text{ with } |\lambda| = F(p; v)$$

(i.e φ isometry at 0 for Poincaré metric of Δ and F).

Remark: (1) + (2) $\Rightarrow$ φ is an isometry: $\varphi^* F =$ the Poincaré metric of Δ (Ahlfors' lemma and Heins' theorem).

6. Kälerianity

Many notions of Kälerianity. Here is the most useful.

Torsion θ of type (2,0) defined by

$$\nabla_X Y - \nabla_Y X = [X, Y] + \theta(X, Y)$$

The decomposition

$$\mathcal{V} \xleftarrow{p_V} T^{1,0}\tilde{M} = \mathcal{V} \oplus \mathcal{H} \xrightarrow{p_H} \mathcal{H}$$

induces

$p_H\theta$ *horizontal part of* θ

$p_V\theta$ *vertical part of* θ

$\theta - p_H\theta - p_V\theta$ *mixed part of* θ.

Proposition 6.1:

(i) $p_V\theta \equiv 0$;

(ii) *mixed part* $\theta - p_H\theta \equiv 0 \Leftrightarrow G = F^2$ *is a hermitian metric;*

(iii) $\theta \equiv 0 \Leftrightarrow G = F^2$ *is a hermitian Kähler metric.*

Definition. F is *Kähler* if for all $H \in \mathcal{H}$

$$\theta(H, \chi) = 0$$

Remark. Other notions of Kählerianity:

- F is *weakly Kähler* if for all $H \in \mathcal{H}$

$$\langle \theta(H, \chi), \chi \rangle = 0$$

Fact: Kobayashi metric for a strongly convex domain is weakly Kähler

- F is *Rund-Kähler* if for all $H, K \in \mathcal{H}$

$$\theta(H, K) = 0$$

Metrics with this property are very easy to handle but very few non hermitian examples.

Conjecture. The Kobayashi metric of a strongly convex domain is Kähler.

7. Geodesics

Identifying $T^{1,0}M$ and TM as real vector bundles, it makes sense to consider F defined on TM.
WARNING: As indicatrix are not strongly convex one cannot apply the theory of real Finsler metrics.

– Lenght of reg. curve $\sigma\colon [a,b] \to M$: $L(\sigma) = \int_a^b F(\dot\sigma(t))dt$

A regular variation of σ with fixed end points:

$$\Sigma\colon (-\epsilon, \epsilon) \times [a,b] \to M$$

such that

$$\begin{cases} \Sigma(0,t) = \sigma(t) \quad \forall t \in [a,b], \\ \Sigma(s,a) = \sigma(a) \quad \text{e} \quad \Sigma(s,b) = \sigma(b), \forall s \\ \sigma_s(t) = \Sigma(s,t) \quad \text{è una curva regolare}, \forall s \\ F\left(\frac{d\sigma_s(t)}{dt}\right) \equiv C_s > 0, \forall s. \end{cases}$$

If $\ell_\Sigma(s) = L(\sigma_s)$ we have

Definition. A regular curve $\sigma\colon [a,b] \to M$ is a *geodesic* for F if for all variations of σ with fixed end points

$$\frac{d\ell_\Sigma}{ds}(0) = 0.$$

Computations of vaiation are based on "exchange of order of derivation" (or better to brackets of vector fields) $\Longrightarrow$ conditions on the torsion of connection.

Notation: If T tangent to the curves σ_s, U tangent to the variation, T^H and U^H the horizontal lifts via horizontal field χ.

Fact: If the metric F is weakly Kähler:

$$\frac{d\ell_\Sigma}{ds}(0) = -\frac{1}{c_0}\operatorname{Re}\int_a^b \langle U^H, \nabla_{T^H+\overline{T^H}} T^H\rangle_{\dot\sigma_0}\, dt.$$

Thus σ is a geodesic if and only if:

$$0 = \nabla_{T^H+\overline{T^H}} T^H =_{loclly} \ddot\sigma^\alpha + \Gamma^\alpha_{;\mu}(\dot\sigma)\dot\sigma^\mu = 0.$$

Consequence:

Theorem 7.1: *If F is a weakly Kähler Finsler metric, then there exists a unique geodesic $\sigma\colon(-\epsilon,\epsilon)\to M$ such that $\sigma(0)=p$ and $\dot\sigma(0)=v$ for all $p\in M$ e $v\in T_pM$.*

Remarks: No use of real convexity; notion of complete complex Finsler metrics, Hopf-Rinow theorem and minimizing distance in the small for geodesics.

– **Totally geodesic complex curves**

Notation: $R_c = \Delta$ (if $c = -2$), $\mathbb{C}$ (if $c = 0$), $\mathbb{P}_1 = \mathbb{C}\cup\{\infty\}$ (if $c = 2$) each with their constant curvature metric.

Definition. A map $\varphi \in Hol(R_c, M)$ for a complex manifold M with a strongly pseudoconvex Finsler metric F is said *c-geodesic complex curve* if it maps geodesics (parametrized by arc lenght) of R_c in geodesics (parametrized by arc lenght) of M.

Remark: Inspired to complex geodesic of invariant metrics.

Theorem 7.2: *Let F be a complete strongly pseudoconvex Finsler metric on cpex manifold M. Then $\forall p \in M$ and $\forall v \in T^{1,0}_p M$ with $F(v) = 1$ there exists a unique c-geodesic complex curve $\varphi \in Hol(R_c, M)$ con $\varphi(0) = p$ e $\varphi'(0) = p$*

$\Leftrightarrow$

F is weakly Kähler, $K_F \equiv 2c$ and $\forall H$

$$\langle \Omega(H, \bar{\chi})\chi, \chi\rangle = \langle \Omega(\chi, \bar{\chi})H, \chi\rangle.$$

Due to Abate-Patrizio (1992). A partial result is due to Pang (1991) who also provided important ideas of the proof.

If F **not** [redacted] is weakly Kähler can prove just existence of infinitesimal isometry between R_c and M. Together with a version of Ahlfors Lemma this shows the following improvement of a result of Faran (1990):

Theorem 7.3: *Let F be a strongly pseudoconvex Finsler metric on a complex manifold M. If $K_F \equiv -4$ and $\forall H$*

$$\langle \Omega(H, \bar{\chi})\chi, \chi \rangle = \langle \Omega(\chi, \bar{\chi})H, \chi \rangle$$

then F is the Kobayashi metric of M.

8. Classification Theory

A strongly pseudoconvex complete Kähler Finsler metric $F: T^{1,0}M \to \mathbb{R}^+$ is called curvature symmetric if $\forall H \in \mathcal{H}$

$$\langle \Omega(H, \bar{\chi})\chi, \chi \rangle = \langle \Omega(\chi, \bar{\chi})H, \chi \rangle. \qquad (8.1)$$

Positive curvature:

Theorem 8.1: *M simply connected complex manifold. If F is complete Kähler-Finsler, curvature symmetric and has constant positive holomorphic curvature $2c > 0$, then (M, F) is biholomorphically isometric to the projective space $\mathbb{P}^n(\mathbb{C})$ endowed with a multiple of the Fubini-Study metric.*

Vanishing curvature:

Theorem 8.2: *M simply connected complex manifold. If F If F is complete Kähler-Finsler, curvature symmetric and has constant vanishing holomorphic curvature, then (M, F) is biholomorphically isometric $\mathbb{C}^n$ endowed with a complex Minkowski metric.*

A technical assumption

The *dual* $(1,1)$-*torsion*:
$T^{1,0}\tilde{M}$-valued form $\hat{\theta} \in \mathcal{X}(\bigwedge^{1,1}\tilde{M} \otimes T^{1,0}\tilde{M})$ s.t.

$$\langle \theta(X,Y), Z\rangle = \langle X, \hat{\theta}(Z,\overline{Y})\rangle$$

for all X, Y, $Z \in T^{1,0}\tilde{M}$.

$\hat{\theta}$ decomposes $\hat{\theta} = \hat{\theta}^{\mathcal{H}} + \hat{\theta}^{\mathcal{V}}$.

Fact 1: F is Kähler iff $\hat{\theta}(H,\bar{\chi}) \equiv 0$
Fact 2: F^2 is Hermitian iff $\hat{\theta}^{\mathcal{V}}(\chi,\overline{K}) = 0 \;\; \forall K \in \mathcal{H}$

Definition F is <u>*tame*</u> if for all $H \in \mathcal{H}$

$$\mathrm{Hess}_G(H) = \mathrm{Re}\big[\langle H,H\rangle + \langle\langle H,H\rangle\rangle\big] \geq \langle \hat{\theta}^{\mathcal{V}}(\chi,\overline{H}), \hat{\theta}^{\mathcal{V}}(\chi,\overline{H})\rangle$$

Note: this is only a punctual requirement on F (i.e., it depends on the derivatives of F along the v directions only, and not on derivatives along the z directions) and it implies strict convexity of F. But

Fact: $g\colon \mathbb{C}^n \to \mathbb{R}^+$ be an Hermitian norm, $f\colon \mathbb{C}^n \to \mathbb{R}^+$ any $(1,1)$-homogeneous function and $\varepsilon << 1$. Then $G = g + \varepsilon f$ is tame $(1,1)$.
i.e: Complex Finsler "near" hermitian <u>are</u> tame

Negative curvature:

Theorem 8.3: *M simply connected complex manifold. If F is complete Kähler-Finsler, curvature symmetric, tame and has constant negative holomorphic curvature, then for any $p \in M$:*

– $\exp_p\colon T_p^{1,0}M \to M$ *is a diffeo in* $C^1(M) \cap C^\infty(M \setminus \{p\})$

– $M \setminus \{p\}$ *is foliated by isometric totally geodesic holomorphic embeddings of the unit disk* Δ *through* p *endowed with (a multiple of) the Poincaré metric*

– *F is (a multiple of) the Kobayashi metric of M*

If ρ *is the distance from* p, *then* $\sigma = (\tanh \rho)^2$ *is an exhaustion of M with the following properties:*

(i) $\sigma \in C^0(M) \cap C^\infty(M \setminus \{p\})$;
(ii) $\sigma \circ \pi \in C^\infty(\check{M})$ $\quad(\pi\colon \check{M} \to M$ blow-up at $p)$;
(iii) σ *is strictly plurisubharmonic on* $M \setminus \{p\}$;
(iv) $(\partial\overline{\partial} \log \sigma)^n = 0$ *on* $M \setminus \{p\}$ *on* $M \setminus \{p\}$;
(v) $\log \sigma(z) = \log \|z\|^2 + O(1)$ *with respect to any coordinate system centered in* p.

In particular M is a Stein manifold.

Theorem 8.4: *M simply connected complex manifold. F is complete Kähler-Finsler, curvature symmetric, tame with constant negative holomorphic curvature. Then:*

– $\exp_p\colon T_p^{1,0}M \to M$ *is of class C^∞ on M) for some $p \in M$*

$$\Longleftrightarrow$$

– *the foliation in isometric disks through p is holomorphic*

$$\Longleftrightarrow$$

$$\Phi(v) = \exp_p\left(\frac{\tanh^{-1}(F(v))}{F(v)}\, v\right)$$

defines a biholomorphism $\Phi\colon I_p(M) \to M$

Theorem 8.5: *M simply connected complex manifold. F is complete Kähler-Finsler, curvature symmetric, tame with constant negative holomorphic curvature. Then:*

F is Berwald in a neighborhood of some $p \in M$

$$\Longleftrightarrow$$

(M, F) is isometrically biholomorphic to the unit ball $\mathbb{B}^n$ with (a multiple of) the Bergmann metric.

A typical application:

Theorem 8.6: *M_1, M_2 simply connected complex manifolds with respectively complete Kähler-Finsler, curvature symmetric, tame metrics F_1, F_2 of constant negative holomorphic curvature. Then a holomorphic map $\Phi\colon M_1 \to M_2$ is biholomorphic $\Longleftrightarrow$*

Φ is an isometry at **one** *point $p \in M$*

(i.e. $F_1(p; v) = F_2(\Phi(p); d\Phi(v))$)

9. Theichmüller metric

$\mathbb{H}^+ \subset \mathbb{C}$ upper half plane
$M \subset L^\infty(\mathbb{H}^+, \mathbb{C})$ the unit ball.

– *Teichmüller metric* $\sigma\colon T^{1,0}M \cong M \times L^\infty(\mathbb{H}^+, \mathbb{C}) \to \mathbb{R}^+$ is the complex Finsler metric on M defined by

$$\sigma(\mu; \nu) = \left\| \frac{|\nu|}{1 - |\mu|^2} \right\|_\infty, \tag{9.1}$$

where $\|\cdot\|_\infty$ is the L^∞ norm and $|\nu(z)|/(1-|\mu(z)|^2) =$ Poincaré length of the tangent vector $\nu(z)$ at $\mu(z) \in \Delta$.

– *Teichmüller distance* on M is integrated distance d_σ of σ:

$$d_\sigma(\mu_1, \mu_2) = \tanh^{-1} \left\| \frac{\mu_1 - \mu_2}{1 - \overline{\mu_1}\mu_2} \right\|_\infty. \tag{9.2}$$

– (M, d_σ) is a complete metric space.

$G = \mathrm{Aut}(\mathbb{H}^+)$ acts on M as a group of lin. isometries via

$$(A, \mu) \mapsto \mu^A = \frac{(\mu \circ A)\bar{A}'}{A'} \tag{9.3}$$

$\forall (A, \mu) \in G \times M$.

If $\Gamma \subset Aut(\mathbb{H}^+)$ is a Fuchsian group (i.e. subgroup acting properly discontinuously on $\mathbb{H}^+$), set

$$L^\infty(\Gamma) = \left\{\mu \in L^\infty(\mathbb{H}^+, \mathbb{C}) \;\middle|\; \mu = \mu^A \quad \forall A \in \Gamma\right\}.$$

The unit ball the closed subspace $L^\infty(\Gamma)$

$$M(\Gamma) = M \cap L^\infty(\Gamma) \tag{9.4}$$

is the space of *Beltrami differentials* relative to Γ.

Teichmüller metric σ and distance d_σ on $M(\Gamma)$ are defined by restriction and $M(\Gamma)$ is a complete Finsler manifold.

Earle-Kra-Krushkal ('94), Abate-Patrizio ('97):

Proposition 9.1: *Let Γ be a Fuchsian group. Then the Teichmüller, Carathéodory and Kobayashi metrics (respectively, distances) of $M(\Gamma)$ coincide. As a consequence if $\varphi\colon \Delta \to M(\Gamma)$ is holomorphic then*

φ isometry at one point for Teichmüller metric iff

φ isometry for Teichmüller metric iff

φ isometry at one point for Teichmüller distance iff
φ isometry for Teichmüller distance

Simple proof as consequence of results due to Harris (79) and Vesentini (81) or corollary of a theorem of Dineen-Timoney-Vigué (85) ($M(\Gamma)$ is a ball!).

$\mathbb{H}^-$: lower half plane in $\mathbb{C}$

B: Banach space $\subset Hol(\mathbb{H}^-, \mathbb{C})$ with norm

$$||\phi||_B = \sup\{|z - \bar{z}|^2 |\phi(z)| \mid z \in \mathbb{H}^-\} < \infty. \quad (9.5)$$

$G = \mathrm{Aut}(\mathbb{H}^+) = \mathrm{Aut}(\mathbb{H}^-)$ acts on B as a group of linear isometries via

$$(A, \phi) \mapsto \phi^A = (\phi \circ A)(A')^2. \quad (9.6)$$

If Γ is a Fuchsian group,

$$B(\Gamma) = \{\phi \in B \mid \phi = \phi^A\}, \quad (9.7)$$

is the subspace of Γ-invariant functions of B.

For $\mu \in M$ there exists a unique quasiconformal homeomorphism w^μ of the Riemann sphere in itself which

– leaves 0, 1, ∞ fixed,

– w^μ is holomorphic on $\mathbb{H}^-$,

– satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on $\mathbb{H}^+$.

The map $\Phi\colon M \to B$ given by $\Phi(\mu) = [w^\mu]$ where $[w] = (w''/w')' - (1/2)(w''/w')^2$ is the Schwarzian derivative, is well-defined (Nehari's theorem).

Bers' description of Teichmüller spaces:

– *universal Teichmüller space*: $T = \Phi(M)$

–*Teichmüller space* of Γ: $T(\Gamma) = \Phi\big(M(\Gamma)\big) \subset B(\Gamma)$

Bers: Topological and holomorphic structures of $T(\Gamma)$ are quotient structure induced by Φ.

Teichmüller metric $\tau_\Gamma \colon T(\Gamma) \times B(\Gamma) \to \mathbb{R}$ is defined using the quotient map Φ:

$$\tau_\Gamma(t;\psi) = \inf\big\{\sigma(\mu;\nu) \bigm| t = \Phi(\mu), d\Phi_\mu(\nu) = \psi\big\}$$

Teichmüller distance d_{τ_Γ} (always complete):

$$d_{\tau_\Gamma}(s,t) =$$

$$\inf\big\{d_\sigma(\alpha,\beta) \bigm| \alpha,\beta \in M(\Gamma), s = \Phi(\alpha), t = \Phi(\beta)\big\}$$

If $M = \Delta/\Gamma$ is compact of genus g

$\Rightarrow$ $T(\Gamma)$ is a bd domain in $\mathbb{C}^{3g-3}$

Royden (71) (and Gardiner in the infinite dimension):

Teichmüller metric (and distance) = Kobayashi metric (and distance)

Consequence: d_{τ_Γ} is the integrated distance of τ_Γ.

What about Carathéodory metric (and distance) on Teichmüller spaces?

Earle (74): Carathéodory metric is complete

Kra (81): Carathéodory and Kobayashi metric agree in many direction.

Krushkal (81,...,85): Carathéodory and Kobayashi metric are different on any $T(\Gamma)$!

Nice properties of finite dimens. Teichmüller spaces:

Proposition 9.2: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space. Then:*

(i) *the Teichmüller distance $\delta_s(t) = d_{\tau_\Gamma}(s,t)$ from a point $s \in T(\Gamma)$ is of class C^1 on $T(\Gamma) \setminus \{s\}$, and the Kobayashi-Teichmüller metric τ_Γ is of class C^1 outside the zero section in $T(\Gamma) \times B(\Gamma)$;*

(ii) *for every $\psi \in B(\Gamma) \cong T_s^{1,0}(T(\Gamma))$ one has*

$$\tau_\Gamma(\psi) = \lim_{h \to 0} \frac{\delta_s(s + h\psi)}{|h|}; \tag{9.8}$$

(iii) *Any two point in $T(\Gamma)$ are joined by a unique geodesic of the Teichmüller metric.*

The regularity of the Kobayashi-Teichmüller metric is due to Royden (71) and presented in details for instance in Gardiner (87). The derivatives of the Kobayashi-Teichmüller distance at a point are computed explicitly in Gardiner (80) and Pang (94) gives proof of (9.8) for any taut domain. Finally (iii) is classical, and the uniqueness follows from Teichmüller uniqueness theorem.

Warning: here finite dimensionality is essential! In infinite dimension no regularity (Li Zhong '96), no uniqueness (Tanigawa '92, Li Zhong '92).

Teichmüller spaces does not negative real curvature and that they are not hyperbolic in any reasonable real sense (Masur-Wolf (1995) for instance), but finite dimensional Teichmüller spaces have a hyperbolic behavior. Reason: negativity of the holomorphic curvature of Kobayashi-Teichmüller metric. Even in the infinite dimensional case (Abate-Patrizio, '98):

Theorem 9.3: *Let Γ be any Fuchsian group. Then the holomorphic curvature of the Kobayashi-Teichmüller metric of $T(\Gamma)$ is identically equal to -4. As a consequence, $T(\Gamma)$ is Kobayashi complete hyperbolic. Furthermore*

a holomorphic curve $\varphi\colon \Delta \to M$ is an isometry at one point between the Poincaré metric on Δ and F $\Longleftrightarrow$ φ is infinitesimal isometry at every point.

Solution of Royden infinitesimal disk conjecture ('86):

Teichmüller disks (which are isometry at one point) are both infinitesimal isometry and complex geodesics curves.

Avalaible differential geometrical techniques do not seem to be suitable to prove that infinitesimal isometry are complex geodesics curves.

By means of a lifting argument of holomorphic disks on Teichmüller spaces (essentially due to Slodkowski '91), Earle-Kra-Krushkal '94 proved:

Theorem 9.4: *Let Γ be a Fuchsian group. Then on $T(\Gamma)$:*

(i) *Full Royden's disk conjecture holds*
(ii) *if $T(\Gamma)$ is finite dimensional, through any point $[\mu] \in T(\Gamma)$ and tangent vector $\psi \in B(\Gamma)$ there is a unique complex geodesic curve $\varphi\colon \Delta \to T(\Gamma)$ such that $\varphi(0) = [\mu]$ and $\varphi'(0)$ is a non-zero multiple of ψ.*

Kobayashi-Teichmüller metric of finite dimensional $T(\Gamma)$ has exactly the same properties concerning existence and uniqueness of complex geodesics as Kobayashi-Carathéodory metric of smooth strictly convex domains in $\mathbb{C}^n$ (Lempert '81).

Infact, even in infinite dimension:

Theorem 9.5: *The indicatrices*

$$I_{[\mu]} = \{\psi \in B(\Gamma) \cong T^{1,0}_{[\mu]}T(\Gamma) \mid \tau_\Gamma(\psi) \leq 1\}$$

of the Kobayashi-Teichmüller metric τ_Γ are convex for every $[\mu] \in T(\Gamma)$.

Inspired by Royden '71, after Graham-Wu '85, Vigue '84', Patrizio '86 and in particular using ideas of Graham '89, (Abate-Patrizio):

Theorem 9.6: *Let Γ be a Fuchsian group so that $T(\Gamma)$ is finite dimensional. A taut connected complex manifold N is biholomorphic to $T(\Gamma)$ iff there exists a holomorphic map $F: N \to T(\Gamma)$ which is an isometry for the Kobayashi metric at one point.*

Other application (C. Castellano 2001):

Theorem 9.7: *Let $T(\Gamma_1), T(\Gamma_2)$ finite dimensional Teichmüller spaces and $p \in T(\Gamma_1), q \in T(\Gamma_2)$. A biholomorphic map $f: \mathbb{B}(p, R) \to \mathbb{B}(q, R)$ of balls with respect to the Teichmüller distance with $f(p) = q$ estends to a biholomorphic map $F: T(\Gamma_1) \to T(\Gamma_2)$*

9. Questions

Many natural questions in Complex Analysis need a more "flexible" Finsler Geometry

For instance the following are interesting problems:

– Kra [Kra] has shown that along special complex geodesics (the so-called abelian Teichmüller disks) the Kobayashi and Carathéodory metrics agree. Is there a geometric characterization of the Teichmüller disks along which the Kobayashi and Carathéodory metrics agree?

– Is there a quasiconformal-free proof of Krushkal's claim that Kobayashi and Carathéodory metrics of Teichmüller metric are different?

– Is it possible a geometric study to Teichmüller metric in infinite dimension?

General Problem 1: Develop the differential geometry of C^1 complex Finsler metrics.

General Problem 2: Develop the differential geometry of complex Finsler metrics in infinite dimension.

Regularization:

M hyperbolic domain in $\mathbb{C}^n$, F its Kobayashi metric,

$I_p = \{v \in \mathbb{C}^n \equiv T_p(M) | F_p(v) < 1\}$ indicatrix at $p \in M$,

h_p hermitian form on $\mathbb{C}^n \equiv T_p(M)$ such that $\{v \in \mathbb{C}^n \equiv T_p(M) | h_p(v) < 1\}$ is the ellipsoid of minimum volume containinig I_p
(Friz John: h_p exists unique.)

Wu metric ('87): $p \mapsto h_p$ is C^0 hermitian metric and it is "essentially" equivalent to Kobayashi metric, in particular biholomorphically invariant

Remark: idea works also on manifolds and may be applied to any Finsler metric.

– what are the regularity properties of Wu metric? If Kob. metric is smooth, is Wu metric a least $C^{1,1}$?

– what are the curvature properties of Wu metric? If Kob. metric neg. curved, is Wu neg. curved?

Positive results in very spcial cases (Kim-Yu '96)