Projective Finsler Geometry

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1) How could we define with Professor Matsumoto a new, special Finsler space, so-called **Douglas space** which is a certain generalization of Berwald space?

2) What kind of rule have the ***Q*-invariants** in the study of Douglas space? Here we obtain: A Finsler space is a Douglas space if and only if the Q^2-invariants depend only on position. From this fact it follows that in a Douglas space the components of Weyl tensor depend only on position, too.

3) How can we find a rectilinear coordinate system in projectively flat Finsler spaces, in which the equation of the geodesics reduces to $d^2x^i/ds^2 = 0$.

4) Finally, we would like to show three "problems" in projective Finsler geometry.

1. PROJECTIVE CHANGE

Let $F^n(M^n, L)$ be an n-dimensional Finsler space where M^n is a connected differentiable manifold of dimension n and $L(x,y)$[1], where $y^i = \dot{x}^i = dx^i/dt$, is the fundamental function defined on the manifold $TM\backslash O$ of non-zero tangent vectors. The system of differential equation for geodesic curves $x^i = x^i(s)$ of F^n with respect to the canonical parameter s (s is the arc-length of the curvature) is given by

$$(1.1) \qquad \frac{d^2x^i}{ds^2} + 2G^i(x,y) = 0.$$

G^i is defined by

$$(1.2) \qquad 2G^i = g^{i\alpha}((\dot{\partial}_\alpha \partial_\beta L^2)y^\beta - \partial_\alpha L^2),$$

where $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ $(\dot{\partial} = \partial/\partial y)$ is the fundamental metric tensor.

Remark 1. ***The Berwald connection coefficients*** *$G^i_j(x,y)$, $G^i_{jk}(x,y)$ can be derived from the function G^i, namely $G^i_j = \dot{\partial}_j G^i$; $G^i_{jk} = \dot{\partial}_k G^i_j$.* ***The Berwald covariant derivative*** *with respect to Berwald connection can be written as*

$$(1.3) \quad T^i_{j;k} = \partial T^i_j/\partial x^k - \dot{\partial}_\alpha T^i_j G^\alpha_k + T^\alpha_j G^i_{\alpha k} - T^i_\alpha G^\alpha_{jk}.$$

[1] $x = (x^1, x^2, ..., x^n)$; $y = (y^1, y^2, ..., y^n)$

If the geodesic is written locally as $x^i = x^i(t)$ for an arbitrary parameter t, then the equation of geodesics are written in the form

$$(1.4) \qquad \frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = \gamma(t)\frac{dx^i}{dt},$$

where $\gamma(t) = (ds^2/dt^2)(ds/dt)$.

Now, we want to give the notion of the projective relation between two Finsler spaces F^n and $\overline{F}^n$. We consider two Finsler spaces $F^n = (M^n, L(x,y))$ and $\overline{F}_n = (M^n, \overline{L}(x,y))$ on a common underlying manifold M^n.

Definition 1. *If any geodesic of F_n coincides with a geodesic of $\overline{F}_n$ as a set of points and vice versa, then the change $L \to \overline{L}$ of the metric is called* ***projective*** *and F^n is said to be* ***projective*** *to $\overline{F}^n$.*

Let $C : x^i = x^i(t)$ be a curve of M^n which is a geodesic of both F^n and $\overline{F}_n$ as a set of points. Then C in (1.4) is in F^n and also in $\overline{F}^n$

$$(1.5) \qquad \frac{d^2x^i}{dt^2} + 2\overline{G}^i(x, \frac{dx}{dt}) = \overline{\gamma}(t)\frac{dx^i}{dt}.$$

Thus, from (1.4) and (1.5) $2\overline{G}^i(x, \frac{dx}{dt}) - 2G^i(x, \frac{dx}{dt}) = (\overline{\gamma} - \gamma)\frac{dx^i}{dt}$ and since this equation must hold for any point x and any direction dx/dt, we have

Theorem 1.1. *(Knebelman, 1927) A Finsler space F^n is projective to another Finsler space $\overline{F}^n$ if and only if there exists a* $(1)p-$ *homogeneous scalar field* $p(x,y)$ *satisfying*

$$\overline{G}^i(x,y) = G^i(x,y) + p(x,y)y^i. \tag{1.6}$$

The scalar field $p(x,y)$ is called the **projective factor** of the projective change under consideration.

2. Projective invariants

From (1.6) we obtain the following equations

$$\overline{G}^i_j = G^i_j + p\delta^i_j + p_j y^i;\ p_j = \dot{\partial}_j p \tag{2.7}$$

$$\overline{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i;\ p_{jk} = \dot{\partial}_k p_j \tag{2.8}$$

$$\overline{G}^i_{jkl} = G^i_{jkl} + p_{jkl}y^i + p_{jk}\delta^i_l + p_{lj}\delta^i_k + p_{kl}\delta^i_j; \tag{2.9}$$

$p_{jkl} = \dot{\partial}_l p_{jk}$.

From the above equation we find that

$$D^i_{jkl} := G^i_{jkl} - \dot{\partial}_l G_{jk} y^i/(n+1) - \\ -(G_{jk}\delta^i_l + G_{lj}\delta^i_k + G_{kl}\delta^i_j)/(n+1) \tag{2.10}$$

where $G_{jk} = G^\alpha_{jk\alpha}$, D^i_{jkl} are components of a tensor field invariant under projective change. This $D-$*tensor* is called the **Douglas projective tensor** (Douglas, 1928).

The important special case of Finsler space is the Berwald space, if the tensor $G^i_{jkl} = 0$.

Thus we obtain

Theorem 2.1. *If a Finsler space is projective to a Berwald space, then its Douglas tensor field vanishes identically.*

Further on we get an another invariant tensor W^i_{jkl} called **the Weyl curvature tensor**, which we can get from Berwald curvature tensor H^i_{jkl}, and the Ricci-type Berwald curvature tensor $H^\alpha_{jk\alpha} = H_{jk}$

(2.11)

$$W^i_{hjk} = H^i_{hjk} + \{\delta^i_n H_{jk} + y^i \dot\partial_h H_{jk} +$$
$$+\delta^i_j \dot\partial_h H_k - \delta^i_h H_{kj} - y^i \dot\partial_h H_{kj} - \delta^i_k \dot\partial_h H_j\}/(n+1)$$

Z. I. Szabó's theorem gave the geometrical meaning of the Weyl tensor:

Theorem 2.2. *(Z.I. Szabó, 1977) Finsler space is of scalar curvature, if and only if the Weyl tensor vanishes indentically.*

Corollary 2.3. *If a Finsler space F^n of scalar curvature is projective to an another Finsler space $\overline{F}_n$, then $\overline{F}_n$ is also of scalar curvature.*

Definition 2. *A Finsler space F^n is said to be* ***of scalar curvature*** *if*

$$H^i_j = h^i_j H, \qquad (2.12)$$

where $h^i_j = \delta^i_j - l^i l_j$, $l^i = y^i/L$; $l_i = \dot\partial_i L$, $H^i_j = H^i_{\alpha\beta j} y^\alpha y^\beta$

It means that the projective mapping (relation) is closed for the Finsler space of scalar curvature.

Thus, there arises an interesting question:

"Which properties are satisfied by Finsler spaces with vanishing Douglas tensor?"

The main purpose of the present lecture is to answer this question in the two-dimension and the n-dimension cases, which is based on four papers:

S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type I, II, IV, *Publ. Math.* Debrecen, 51 (1997),385 – 406, 53(1998),423 – 438, 56(2000), 213 – 221.

S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type III, Kluwer Academic Publishers, 2000, 89 – 94.

The notion of the locally Minkowski space is very important for us:

Definition 3. *A Finsler space with a fundamental function $L(x, y)$ is called **locally Minkowski**, if there exists a coordinate system (x^i) in which $L(x, y)$ is a function of y^i only. Such a coordinate system (x^i) is called adapted.*

Definition 4. *A Finsler space is called **projectively flat**, if it has a covering by coordinate neighbourhoods in which it is projective to a locally Minkowski space.*

Theorem 2.4. *The Douglas tensor D and the Weyl tensor W vanish identically in a projectively flat Finsler space.*

3. Finsler spaces of Douglas type

Definition 5. *A Finsler space is said to be of* ***Douglas type*** *or* ***a Douglas space****, if* $D^{ij} = G^i y^j - G^j y^i$ *are homogeneous polinomials in* (y^i) *of degree three.*

We are concerned with a two-dimensional Finsler space F^2 with a local coordinate system $(x^1, x^2) = (x, y)$ and we put $(y^1, y^2) = (p, q)$. Let us take x as a parameter of curves and use the notation as follows $y' = dy/dx$, and $y'' = d^2y/dx^2$.

Theorem 3.1. *A two-dimensional Finsler space is a Douglas space if and only if in a local coordinate system* (x, y)*, the right-hand side of* $f(x, y, y')$ *of the equation of geodesics* $y'' = f(x, y, y')$ *is a polynomial in* y' *of degree at most three.*

We treat a Finsler space F^n with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$. F^n is by definition a Douglas space if and only if

$$\dot\partial_k \dot\partial_j \dot\partial_i \dot\partial_h (G^l y^m - G^m y^l) = 0. \tag{3.13}$$

After some calculations and considerations from (3.13) we get that the Douglas tensor vanishes identically, and conversely, if the Douglas tensor of an F^n vanishes identically, then F^n is a Douglas space. Therefore we can state the following theorem.

Theorem 3.2. *A Finsler space is of Douglas type if and only if the Douglas tensor vanishes identically.*

Definition 6. *A Finsler space is called* ***a Landsberg space*** *if the condition* $y_\alpha G^\alpha_{jkl} = 0$ *holds.*

Theorem 3.3. *If a Finsler space* F^n $(n \geq 2)$ *is a Landsberg space and a Douglas space, then it is a Berwald space. Conversely, a Berwald space is a Landsberg space and a Douglas space.*

The Douglas tensor is projective invariant. Hence the following theorem is true.

Theorem 3.4. *If a Finsler space is projectively related to a Douglas space then it is also a Douglas space.*

Example 1. *The family of solutions of a second order linear differential equation*

$$y'' + P(x)y' + Q(x)y = R(x) \tag{3.14}$$

coincides with the family of geodesics of the two-dimensional Finsler space F^2 *with the metric*

$$L(x, y, p, q) = \frac{1}{p}\exp\left(\int P dx\right)[(2R - Qy)yp^2 + q^2]. \tag{3.15}$$

Consequently, this F^2 is a Douglas space.

Example 2. *Let the second order differential equation $y'' = (y')^2 + 1$ be given. The solution of this equation: $y = c_1 - \log|\cos(x + c_2)|$ (c_1 and c_2 are arbitrary constants) are the geodesics of a two-dimensional Finsler space F^2 with metric function*

(3.16)

$$L(x, y, p, q) = q \tan^{-1} \frac{q}{p} - p \log \sqrt{1 + (\frac{p}{q})^2} - xq.$$

It is easy to show that this F^2 is certainly **not a Berwald space**, however, we have $2(G^1 q - G^2 p) = p(p^2 + q^2)$, which implies again that F^2 is a Douglas space.

Example 3. *Consider a Finsler space $F^n = (M^n, L(\alpha, \beta))$ with (α, β)-**metric** where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form. The Riemannianian space $R^n = (M^n, \alpha)$ is said to associate with F^n. **The Randers metric** $L = \alpha + \beta$ and the **Kropina metric** $L = \frac{\alpha^2}{\beta}$ have played a central rule in the theory of (α, β)-metrics. In R^n we have the Christoffel symbols $\gamma^i_{jk}(x)$ and the covariant derivation (,) with respect to γ^i_{jk}. We shall use the symbols as follows:*

$$r_{ij} = \tfrac{1}{2}(b_{i,j} + b_{j,i}), \quad s_{ij} = \tfrac{1}{2}(b_{i,j} - b_{j,i}), \quad s^i_j = a^{i\alpha} s_{\alpha j};$$
$$s_j = b_\alpha s^\alpha_j; \quad b^2 = a^{\alpha\beta} b_\alpha b_\beta; \quad s_{ij} = (\partial_j b_i - \partial_i b_j)/2.$$

Theorem 3.5. *A Randers space is of Douglas type if and only if* $s_{ij} = 0$, *that is,* β *is a closed form.*

Theorem 3.6. *A Kropina space* F^n $(n > 2)$ *with* $b^2 \neq 0$ *is of Douglas type if and only if*

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{3.17}$$

is satisfied.

Theorem 3.7. *All two-dimensional Kropina spaces are Douglas spaces.*

4. Q-INVARIANTS

A Finsler space F^n is said to be projectively related or projective to another Finsler space $\overline{F}_n$, if any geodesic of F^n is a geodesic of $\overline{F}_n$ and vica-versa. The condition under which it holds is written in the following form:

$$\overline{G}^i = G^i + py^i, \tag{4.18}$$

where $p = p(x, y)$ is scalar function.

The change $F^n \to \overline{F}^n$ is called projective. From (4.18) we have

$$\overline{G}^h_i = G^h_i + p_i y^h + p\delta^h_i; \; p_i = \dot{\partial}_i p, \tag{4.19}$$

$$\overline{G}^h_{ij} = G^h_{ij} + p_{ij}y^h + p_i\delta^h_j + p_j\delta^h_i; \; p_{ij} = \dot{\partial}_j p_i. \tag{4.20}$$

If we take $G = G^\alpha_\alpha$, then (4.18) and (4.19) give $\overline{G} = G + (n+1)p$,

$$\overline{G}^h - \frac{\overline{G}}{n+1}y^h = G^h - \frac{G}{n+1}y^h \tag{4.21}$$

which gives rise to a projective Q^0**-invariant**

$$Q^h = G^h - \frac{G}{n+1}y^h; \; G = G^\alpha_\alpha. \tag{4.22}$$

Taking $Q^h_i = \dot{\partial}_i Q^h$ and $Q^h_{ij} = \dot{\partial}_j Q^h_i$, which are the Q^1**-invariant** and Q^2**-invariant**, respectively, we have

$$Q^h_{ij} = G^h_{ij} - \frac{1}{n+1}(G_{ij}y^h + G^\alpha_{\alpha i}\delta^h_j + G^\alpha_{\alpha j}\delta^h_i). \tag{4.23}$$

Finally, we get a remarkable expression of the Douglas tensor as follows:

$$\dot\partial_k \dot\partial_j \dot\partial_i Q^h = D^h_{ijk}. \tag{4.24}$$

It follows from (4.24) that

Theorem 4.1. *A Finsler space is of Douglas type if and only if Q^h of* (4.22) *are homogeneous polynomials in (y^i) of degree two.*

Let F^n be a Douglas space. Then Q^h_{ij} of (4.23) are functions of position (x) alone and we may take

$$Q^h_{ij}(x) y^i y^j = 2G^h - \frac{2}{n+1} G^\alpha_\alpha y^h, \tag{4.25}$$

which implies

$$2D^{hk} = (Q^h_{ij}(x) y^i y^j) y^k - (Q^k_{ij}(x) y^i y^j) y^h. \tag{4.26}$$

Theorem 4.2. *For a Douglas space, $D^{hk} = G^h y^k - G^k y^h$ are homogeneous polynomials in (y^i) of degree three as written in the form* (4.26).

In particular, for a two-dimensional Douglas space, the equation of geodesics
$y'' = f(x, y, y')$ is written

$$(4.27)\qquad y'' = A(x,y)(y')^3 + B(x,y)(y')^2 + C(x,y)(y') + D(x,y),$$

where $A(x,y) = G^1_{22}$; $B(x,y) = 2G^1_{12} - G^2_{22}$; $C(x,y) = G^1_{11} - 2G^2_{12}$; $D(x,y) = -G^2_{11}$.

It is more convenient to write it in the following form

$$(4.28)\qquad y'' = Q^1_{22}(y')^3 + (2Q^1_{12} - Q^2_{22})(y')^2 + (Q^1_{11} - 2Q^2_{12})y' - Q^2_{11}.$$

Since the Q-invariants $Q^h_{ij}(x)$ will play various essentiall rules in the theory of Douglas spaces, we give the following definition.

Definition 7. *The set* $\{Q^h_{ij}(x)\}$ *is called the characteristic of Douglas space. From* Q^2*-invariants we obtain the following* Q^3***-invariants*** *in the Douglas space*

$$(4.29)\quad Q^h_{ijk}(x) = \partial_k Q^h_{ij} + Q^\alpha_{ij} Q^h_{\alpha k} - \partial_j Q^h_{ik} - Q^\alpha_{ik} Q^h_{\alpha j}.$$

From (4.29) we get the following π-tensor which is invariant under projective relation

$$(4.30)\qquad \pi^h_{ijk}(x) = Q^h_{ijk} + \frac{1}{n-1}(\delta^h_j Q_{ik} - \delta^h_{ik} Q_{ij}).$$

Theorem 4.3. *The Weyl tensor W coincides with the π-tensor the components of which are written in terms of Q^3-invariants as in (4.29), where $Q_{ij} = Q^{\alpha}_{ij\alpha}$.*

Theorem 4.4. *For a Douglas space the components of the Weyl tensor are functions of positions alone.*

From Q^3-invariants we can define the following tensor

$$\pi_{ijk}(x) = \partial_k Q_{ij} - Q^{\alpha}_{ij} Q_{\alpha k} - \partial_j Q_{ik} + Q^{\alpha}_{ik} Q_{\alpha j} \tag{4.31}$$

which is also invariant under projective relation.

Consequently, by equation (4.24) and (4.30) we can state that both projective invariant tensor, the Douglas tensor and Weyl tensor are obtained from the invariants Q^i. In a Douglas space Q^i_{jk} are functions of the position (x^i) alone, and so is the Weyl tensor.

5. Rectilinear coordinate system

If a Finsler space $F^n = (M^n, L(x,y)$ **is locally Minkowski space**, then we have a covering of M^n by the domains of **adapted coordinate systems** $(U, (x^i))$ in which L in a function of y^i alone, the quantities G^i vanish in U and the equation of geodesics reduces to $d^2x^i/ds^2 = 0$. Therefore, any geodesic is given by n linear equation $x^i = x_0^i + sv_0^i$ on the arc-length s with $2n$ constants x_0^i and v^i.

~~Theorem~~ Definition 5.1. *A Finsler space $F^n = (M^n, L(x,y))$ is said to be equipped* ***with rectilinear extremals****, if M^n is covered by coordinate neighborhood $(U, (x^i))$ in which any geodesic is represented by n linear equations $x^i = x_0^i + ta^i$ of a parameter t, or $n-1$ linearly independent linear equations $a_i^\alpha(x^i - x_0^i) = 0$, $\alpha = 1, 2, ..., n-1$.*

Therefore, a locally Minkowski space is with rectilinear extremals, and if F^n is projective to a locally Minkowski space $\overline{F}^n$ $(\overline{G}^i = 0)$, then any geodesic is represented in an adapted coordinate system (U, x^i) of $\overline{F}^n$ by $x^i = x_0^i + \bar{s}v_i^0$ as above, so that F^n is with rectilinear extremals. This (U, x^i) coordinate system is said to be a rectilinear coordinate system in F^n.

Theorem 5.2. *A Finsler space is equipped with rectilinear extremals if and only if it is projectively flat.*

Let us consider a projectively flat space F^n. As it has been mentioned, F^n has a covering by rectilinear coordinate neighbourhoods in which there exists a function $p(x,y)$ satisfying $G^i = py^i$, that is, $D^{ij} = G^i y^j - G^j y^i$. Consequently, F^n is a kind of Douglas space. From $G^i = py^i$ we obtain $Q^h = 0$, conversely $Q^h = 0$ leads to $G^h = Gy^h/(n+1)$. Therefore we have

Theorem 5.3. *A projectively flat Finsler space is a Douglas space.*

A coordinate system (x^i) of a projectively flat space is rectilinear if and only if the characteristic Q^h vanishes identically in (x^i).

Theorem 5.4. *A rectilinear coordinate system $\overline{x}^a$ of a projectively flat space is obtained from any coordinate system (x^i) by the differential equations*

$$\partial_i \overline{x}^a = \overline{X}^a_i \tag{5.32}$$

$$\partial_j \overline{X}^a_i = Q^k_{ij}\overline{X}^a_k + Y_i \overline{X}^a_j + Y_j \overline{X}^a_i \tag{5.33}$$

$$\partial_j Y_i = Y_i Y_j + Q^k_{ij} Y_k + Q_{ij}/(n-1), \tag{5.34}$$

where $Y_i := \frac{1}{n+1}\overline{X}^a_{jr}\underline{X}^r_a$, $\underline{X}^i_a = \partial x^i/\partial \overline{x}^a$, $\overline{X}^a_{jk} = \partial_k \overline{X}^a_j$.

Therefore we obtain the complete system of differential equations (5.32), (5.33), (5.34) for the functions $(\overline{x}^a, \overline{X}_i^a, Y_i)$. This system is completely integrable if and only if the tensors π_{ijk}^h and π_{ijk} vanish identically. Thus, we obtain a set of solution $(\overline{x}^a, \overline{X}_i^a, Y_i)$, and we can show that $(\overline{x}^a)$ is certainly a rectilinear coordinate system.

In the case $n > 2$, $\pi_{ijk}^h = 0$ leads to $\pi_{ijk} = 0$, in the case $n = 2$ $\pi_{ijk}^h = 0$ is only an identity.

Summarizing all the above mentioned facts, we can state the following theorem.

Theorem 5.5. *A Finsler space is projectively flat if and only if F^n is Douglas space and its characteristic satisfies:*

(1)$n > 2$: $\pi_{ijk}^h = 0$

(2)$n = 2$: $\pi_{ijk} = 0$, and a rectilinear coordinate system $\overline{x}^a$ is obtained by solving the system of differential equations (5.32), (5.33), (5.34).

Example 1. *Assume that a two-dimensional Finsler space F^2 in a domain of the (x, y)-plane if the geodesic given by the equation $y'' = f(x, y)$ where y' is not contained. Then $\pi_{112} = -\partial_y\partial_y f$, $\pi_{212} = 0$. Consequently F^2 is projectively flat if and only if $f(x, y)$ is linear in y, $f(x, y) = A(x)y + B(x)$, that is, there exists a rectilinear coordinate system $(\overline{x}, \overline{y})$ in F^2, where the equation of geodesics is $\overline{y}'' = 0$.*

6. Three examples for projectively flat Finsler spaces

We have a remarkable interesting two (α, β)-metrics, one is the Randers metric $L = \alpha + \beta)$ and the other is the Kropina metric $L = \alpha^2/\beta)$ where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. A Randers space is a Finsler space with the Randers metric, and a Kropina space is a Finsler space with the Kropina metric.

Example 1.

Theorem 6.1. *(Matsumoto, 1991) A Randers space is projectively flat if and only if its associated Riemannianian space with the metric α is projectively flat and the change $\alpha \to \alpha + \beta$ is projective.*

Example 2.

Theorem 6.2. *A Kropina space of dimension $n > 2$ is projectively flat, if and only if $s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i)$ and $\pi^{\circ h}_{ijk} + K^h_{ijk} + (\delta^h_j K_{ik} - \delta^h_k K_{ij})/(n-1) = 0$, where π° is the Weyl tensor of the associate Riemannianian space and*

$$K^h_{ijk} = K^h_{ij,k} + K^\alpha_{ij} K^h_{\alpha k} - K^h_{ij,k} - K^\alpha_{ij} K^h_{\alpha j};$$

$$K^h_{ij} = Q^h_{ij} - Q^{\circ h}_{ij}; \quad K_{ij} = K^\alpha_{ij\alpha}.$$

Example 3. *Now we consider the* n *dimensional Berwald spaces* $B(n)$ *and the* n*-dimensional projectively flat Finsler spaces* $P(n)$.

What is $B(n) \bigcap P(n)$*?*

Theorem 6.3. *A Finsler space* F^n *is projectively flat Berwald space if and only if it belongs to one of the following classes*

(1) $n \geq 3$

a) locally Minkowski spaces,

b) Riemannian spaces of constant curvature,

(2) $n = 2$

a) locally Minkowski spaces,

b) Riemannian spaces of constant curvature,

c) spaces F^2 *with* $L = \beta^2/\gamma$, *where* β *and* γ *are* 1*-forms.*

7. Three questions in projective Finsler geometry

I.W. Roxburgh and his collaborators have studied Finsler spaces which are projective to Riemannian as for the theory of a space time and gravitation (General Relativity and Gravitation 18 (1986), 849-859; 23(1991), 1071-1080).

From this researches we can formulate the following **first problem**:

"Determine all the Finsler spaces which have common geodesics with some Riemannian space, that is, determine all the Finsler spaces projective to a Riemannian space". (S. Bácsó, 1993).

Hence, if F^n is projective to a Berwald space, then F^n is of Douglas type. Thus, denoting by $D(n)$ the set of all n-dimensional Douglas spaces and by $pB(n)$ the set of all Finsler spaces which are projective to a Berwald space, we have the important subset $pB(n)$ of $D(n)$. If we denote by $pR(n)$ the set of all n-dimensional Finsler spaces which are projective to a Riemannian space then we have the relation as follows: $pR(n) \subset pB(n) \subset D(n)$.

From Szabó 's result, which stated that any Berwald connection is Riemannian metrizable, $pR(n) = pB(n)$ follows for a positive definite Finsler metric.

The **second problem** is as follows:

"To find the tensorial characterisation of projectivity to Berwald spaces $pB(n)$" (M. Matsumoto, 1998).

Thus the **third problem** which is a consequence of the second problem became important:

"Is there any Douglas space which is not projective to Berwald spaces$(pB(n) = D(n)?)$ (Z. Shen, 2000).

Finally, we give a Douglas space (which is not Berwald) and a Berwald space which have not common geodesics. Let us consider a Randers metric $L = \alpha + \beta$, which is a Douglas metric and not Berwald metric (then $s_{ij} = 0$ and $b_{i,j} \neq 0$), and a Kropina metric $L = \alpha^2 + \beta$, which is a Berwald space (then r_{ij} is proportional to a_{ij}, that is, $r_{ij} = u(x)a_{ij}$ for some $u(x)$, and $s_{ij} = (b_i s_j - b_j s_i)/b^2$ $(\alpha^2 = a_{ij}(x)y^i y^j$, $\beta = b_i(x)y^i)$. Hence, the equations

$$(7.35) \qquad b_{i,j} - b_{j,i} = 0$$

$$b_{i,j} + b_{j,i} = 2u(x)a_{ij}$$

$$b_{i,j} - b_{j,i} = \tfrac{1}{2}(s_i b_j - s_j b_i)$$

are realized at the same time.

By this equation we obtain

$$(7.36) \qquad b_{i,j} = u(x)a_{ij}$$

The equation (7.36) is completely integrable equation in a Rieman space of nonzero-constant curvature. Then $u(x) \neq 0$.

So we obtain two Finsler spaces: **a Randers space** $F^n = (M^n, L = \alpha + \beta)$ **of Douglas type** (which is not Berwald), and **a Kropina space** $\overline{F}^n = (M^n, \overline{L} = \alpha^2/\beta)$ **of Berwald type**, which are induced by the same Riemannianian metric α, and a one-form β.

It is known that a Randers space $F^n = (M^n, L = \alpha + \beta)$ is projective to a Kropina space $F^n = (M^n, \overline{L} = \alpha^2 + \beta)$, then $b_{i,j} = 0$ independently of the dimension n.

By (7.36) we have $b_{i,j} \neq 0$ in F^n and $\overline{F}^n$, so the Douglas-Randers space F^n given above is not projective to the Berwald-Kropina space $\overline{F}^n$.