Holonomy Structures in Finsler Geometry

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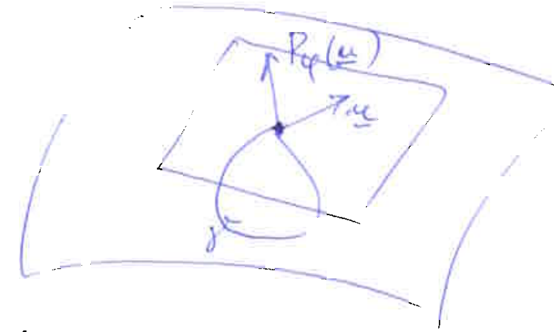
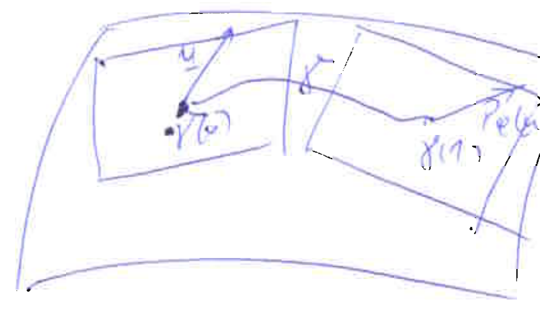
- – holonomy groups in Riemannian geometry
- – Finsler metrics
- – connections in Finsler geometry
- – holonomy of homogeneous connections
- – mixed holonomy of the Finsler vector bundle
- – special Finsler spaces: Minkowski, Berwald, Landsberg spaces

Holonomy groups in Riemannian geometry



M^n : connected and simply connected n-manifold
g : Riemannian metric on M
parallel transport along curves:
for each (piecewise C^1) curve $\gamma : [0,1] \to M$, there is associated a linear mapping $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$
an isometry of vector spaces
$P_{\bar\gamma} = P_\gamma^{-1}$ and $P_{\gamma_2\gamma_1} = P_{\gamma_2} \circ \ P_{\gamma_1}$ where $\bar\gamma$ is the path defined by $\bar\gamma(t) = \gamma(1-t)$
and $\gamma_2\gamma_1$ is defined only when $\gamma_1(1) = \gamma_2(0)$:

$$\gamma_2\gamma_1(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$



The holonomy group H_x : for any $x \in M$, the set of linear transformations of the form P_γ where $\gamma(0) = \gamma(1) = x$ is a subgroup $H_x \subset O(T_x M)$
for any other point $y \in M$, we have $H_y = P_\gamma H_x P_{\bar{\gamma}}$
where $\gamma : [0,1] \to M$ satisfies $\gamma(0) = x$ and $\gamma(1) = y$.
M is simply connected : H_x is connected and hence is a closed Lie subgroup of $\mathrm{SO}(T_x M)$.

Ambrose-Palais theorem: The holonomy algebra $\mathfrak{h}_x$ is spanned by all elements of the form $P_{\varphi^{-1}} \circ R(P_\varphi(u), P_\varphi(v))$ involving the curvature mappings $R_{u,v}$ for all vectors fields u and v and the parallel translations P_φ along curves φ, $\varphi(0) = x$.

De Rham Theorem: if there is a splitting $T_xM = V_1 \oplus V_2$ which remains invariant under all the action of H_x, then the metric g is locally a product metric in the following sense:
The metric g can be written as a sum of the form $g = g_1 + g_2$ in such a way that, for every point $y \in M$ there exists a neighborhood U of y, a coordinate chart $(x_1, x_2) : U \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and metrics $\overline{g}_i$ on $\mathbb{R}^{d_i}$ so that $g_i = x_i^*(\overline{g}_i)$.

In the reducible case the holonomy group H_x is a direct product of the form $H_x^1 \times H_x^2$ where $H_x^i \subset \mathrm{SO}(V_i)$.
For each of the factor groups H_x^i, there is a submanifold $M_i \subset M$ so that $T_xM_i = V_i$ and so that H_x^i is the holonomy of the Riemannian metric g_i on M_i.

Theorem. *(Berger, 1955) Suppose that g is a Riemannian metric on a connected and simply connected n-manifold M and that the holonomy H_x acts irreducibly on T_xM for some (and hence every) $x \in M$. Then either (M, g) is locally isometric to an irreducible Riemannian symmetric space or else there is an isometry $\iota : T_xM \to \mathbb{R}^n$ so that $H = \iota H_x \iota^{-1}$ is one of the subgroups of SO(n) in the following table.*

Subgroup	Conditions	Geometrical Type
SO(n)	any n	generic metric
U(m)	$n = 2m > 2$	Kähler
SU(m)	$n = 2m > 2$	Ricci-flat Kähler
Sp(m)Sp(1)	$n = 4m > 4$	Quaternionic Kähler
Sp(m)	$n = 4m > 4$	Hyperkähler
G_2	$n = 7$	Associative
Spin(7)	$n = 8$	Cayley

In Finsler geometry, the notion of holonomy admits to introduce several types of holonomy groups at different levels. It is not clear yet what the most adequate is. A detailed study on the holonomy group of homogeneous connection is given Barthel, and in other aspects by Okada. These are, however, not linear groups. A strong result is given by Grangier for the reducibility of the mixed holonomy group.

The notion of a Finsler metric

Approach I: $\forall p \in M \quad L_p : T_pM \to R^+$ norm

- $L_p(u) \geq 0 \qquad = 0 \Longleftrightarrow u = 0$
- $L_p(\lambda(u)) = \lambda L_p(u) \qquad \lambda > 0$ positively homogeneous
- $L_p(u + v) \leq L_p(u) + L_p(v)$ convexity
- $L : TM \to R^+$ is of class C^1
- $L : TM \setminus \{0\} \to R^+$ is of class C^2
- $L_p(-u) = L_p(u)$ symmetrical/ reversible

indicatrix: $I_p = \{u \in T_pM \mid L_p(u) = 1\}$

Approach II: variational problem

$$\int_a^b L(x(t), \dot{x}(t))dt \longrightarrow \text{Euler-Lagrange equations}$$

. $\uparrow$ positively homogenous

Riemannian case: $L(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$

Finslerian case: $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$

Approach III: $d : M \times M \to R^+$ is a metric
$v \in T_pM; c : [0,1] \to M$ with $c(0) = p$, $\dot{c}(0) = v$

$$L_p(v) = \lim_{t \to 0} \frac{d(p, c(t))}{t}$$

A function $L\colon TM \to R$ is called a **Finsler fundamental function** in a tangent bundle τ_M if

1. $L(u) > 0 \;\; \forall u \in TM, \;\; u \neq 0$

2. $L(\lambda u) = \lambda L(u) \;\; \forall \lambda \in R^+, u \in TM$

3. L is smooth except on the zero section

4. $g_{ij}(x,y) = \dfrac{\partial^2(1/2L^2)}{\partial y^i \partial y^j}(x,y)$ is positive definite for $(x,y) \neq 0$.

The last assumption implies that the indicatrix

$$I_x = \{z \in T_xM = \pi^{-1}(x) \quad | \quad L(z) = 1\}$$

at each point $x \in M$ is convex. The indicatrix bundle $\mathfrak{I}\tau_M = (ITM, \overline{\pi}, M, S^{m-1})$ of a Finslerian tangent bundle (τ_M, L) is formed with the indicatrices I_x as fibres.

Roughly speaking, a Finsler fundamental function L_p at $p \in M$ gives a norm in the tangent space T_pM. Thus a Finsler space (M, L) is a manifold M endowed with a Finsler fundamental function L.

Example 1: Funk metric

$\Omega \subset R^n$ strictly convex

$$d(p,q) = \ln \frac{|z-p|}{|z-q|}$$

$$p + \frac{y}{L(y)} \in \partial\Omega$$

$$B^n = \Omega; \quad L(y) = \frac{\sqrt{|y|^2 - (|p|^2|y|^2 - (p,y)^2)} + (p,y)}{1-|p|^2}$$

— projectively flat

— constant negative curvature -1

— non–reversible

— Randers metric

Example 2: Hilbert metric

$$\tilde{d}(p,q) = \frac{1}{2}(d(p,q) + d(q,p)) = \frac{1}{2}\left|\ln\left(\frac{|z-p|}{|z-q|} : \frac{|v-p|}{|v-q|}\right)\right|$$

Example 3: slope metric with time measure

surface in R^3: $x^3 = f(x^1, x^2)$

standard Riemannian metric:

$\alpha(x,y) = \sqrt{(y^1)^2 + (y^2)^2 + (\partial_1 f y^1 + \partial_2 f y^2)^2}$

horizontal speed v_0; $v = v_0(1 - a\cos\omega)$

$$T = \int_{\tau_1}^{\tau_2} \frac{\alpha^2}{v_0\alpha - a\beta} dt$$

$$L(x,y) = \frac{\alpha^2(x,y)}{v_0\alpha(x,y) - a\beta(x,y)}$$

Example 4: Katok's example (1973), W. Ziller (1982): Randers space

S^2; standard Riemannian metric α
Φ_t: one parameter group of rotations leaving the north & south poles invariant
X: Killing vector field
β: Killing form

$$L_\varepsilon(x,y) = \alpha(x,y) + \varepsilon\beta(x,y)$$

Theorem: For any irrational ε a curve γ is a closed geodesic of L_ε if and only if γ is a closed geodesic of α and invariant with respect to Φ_t.

Properties:

– the length of the two closed geodesics: $\dfrac{2\pi}{1+\varepsilon}$; $\dfrac{2\pi}{1-\varepsilon}$

– L_ε is a Finsler metric $\Longleftrightarrow$ $|\varepsilon| < 1$

The bundles

$\tau_M = (TM, M, \pi)$ denotes the tangent bundle, a base manifold M, a total space TM ($2n$ dimensional), and a projection $\pi\colon TM \to M$. $Sec\,\tau_M$ denotes the set of all differentiable sections of τ_M.

V_zTM is the kernel of $(d\pi)_z\colon T_zTM \to T_{\pi(z)}M$. Then the vertical bundle $V\tau_M = (VTM, TM, \pi_V)$ is a subbundle of τ_{TM}, and isomorphic to $\pi^*(\tau_M) = (TM \times_M TM, TM, pr_1)$. The latter isomorphism is described as follows: $\varepsilon\colon \pi^*(\tau_M) \to V\tau_M \quad \varepsilon(z_1, z_2) =$ the tangent of the curve $z_1 + tz_2$ at 0, where $z_1, z_2 \in TM$ with the property $\pi(z_1) = \pi(z_2)$.

By **a connection** of τ_M we mean a splitting $H\colon \pi^*(\tau_M) \to \tau_{TM}$ of the next short exact sequence

$$0 \to V\tau_M \xrightarrow{\iota} \tau_{TM} \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_M) \to 0 \qquad (1)$$

where $\widetilde{d\pi}\colon \tau_{TM} \to \pi^*(\tau_M)$ is given by $\widetilde{d\pi}(A) = (\pi_{TM}(A), d\pi(A))$ for $A \in TTM$. H is also called a horizontal map, and its images

$$H_zTM = ImH|_{\{z\}\times T_{\pi(z)}M}$$

are the horizontal subspaces which are complementary to the vertical subspaces: $\tau_{TM} = V\tau_M \oplus H\tau_M$.
Let $\varphi\colon I \to M$ a curve in the base space.
A vector field $Y \in \mathfrak{X}(M)$ is called parallel along φ if $dY(\dot{\varphi})$ are horizontal vectors, where $\dot{\varphi}$ denotes the tangent curve of φ. This means that $H(Y \circ \varphi, \dot{\varphi}) = dY(\dot{\varphi})$.

The parallel translation of z along φ is denoted by $P_\varphi(t, z)$.
The covariant derivation $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is given as $\nabla_X Y = \alpha(v(dY(X)))$ for all $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$, where $\alpha = pr_2 \circ \varepsilon^{-1}\colon V\tau_M \to \tau_M$.

Denote $\mu_t\colon TM \to TM$ the multiplication by $t \in R$ in the fibres of τ_M. It is said that a horizontal map satisfies the homogeneity condition if

$$H(\mu_t(z), v) = d\mu_t(H(z, v)) \tag{2}$$

holds for all $z \in TM$, $v \in TM$ and $t \in R$. If the differentiability of H is not assumed at the zero vectors of τ_M, and H satisfies (2), we speak of a **homogeneous connection** (nonlinear connection). When H satisfies (2) and differentiable anywhere, then we get a **linear connection.**

connection – parallel transport – covariant derivation

$TTM, VTM, \pi_M^*(\tau_M)$

Berwald, Cartan, Chern–Rund, Barthel

geodesics – paths

$(M, L); \quad L(x, y)$
$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ positive definite; Riemannian metric in VTM
$G^i = \frac{1}{4}g^{ik}(y^r\dot{\partial}_k\partial_r L^2 - \partial_k L^2)$; geodesic parameter
$G^i_j = \dot{\partial}_j G^i$; — homogeneous (nonlinear) connection on M
$G^i_{jk} = \dot{\partial}_j G^i_k$; — connection in VTM
(VTM, g) — Riemannian metric
(VTM, ∇^B, N) — Berwald connection pair

The pull back bundle $\pi^*(\tau_M) = (TM \times_M TM, TM, pr_1)$ is called now a **Finsler vector bundle.**
$g_{ij}(x, y)$ gives a Riemannian metric in the Finsler vector bundle.
Definition. *A pair* (∇^F, H) *is called* **a Finsler pair connection** *where* ∇^F *is a linear connection of* $\pi^*(\tau_M)$, *and* H *is a homogeneous connection of* τ_M.

On a Finsler manifold there are known several important Finsler pair connection such as of Cartan, Berwald, Rund, etc.

The **holonomy group for homogeneous connections** is defined as usual for linear connections: It is the group at the point x generated by the parallel translations along all loops at x. This is not a subgroup of the linear group but gives a subgroup of of the group of all invertable positively homogeneous differentiable map of the fibre. In general it is neither infinite dimensional diffeomorphism group nor Lie group.

The holonomy groups of the homogeneous connection H, denoted by G^h_x at $x \in M$:

$$G^h_x = \{ P_\varphi : T_x M \to T_x M \,|\, \varphi : [0,1] \to M, \varphi(0) = \varphi(1) = x \}$$

It was defined and investigated in details by W. Barthel (1963).

G_x^h is not a linear group except in the case of Berwald spaces, however, it is a subgroup of the locally Banach topological group B_x consisting of positively homogeneous bijective C^∞ mappings on $TM \setminus \{0\}$.

$$B_x = \{\,\Phi : T_xM \to T_xM \,|\, \Phi \text{ is pos.hom.: } \Phi(\lambda v) = \lambda \Phi(v), \lambda > 0\}$$

For a Finsler manifold (M, L) the parallel translation of the Barthel homogeneous connection N preserves the length of vectors, i.e. the homogeneous holonomy group G_x^h is a subgroup of the norm-preserving positively homogeneous transformations

$$\{T\colon T_xM \to T_xM \mid \text{pos.hom.}, \ L \circ T = L, \ C^\infty \text{on}\, T_xM \setminus \{0\}\}$$

Holonomy algebra

$A, B : T_xM \to T_xM$ positively homogeneous mappings
Lie bracket: $[A, B](v) = dB_v(A(v)) - dA_v(B(v))$
covariant derivative:
For $A_t : [0, 1] \to \mathrm{End}(T_xM)$ (1,1)-tensor field along a curve γ_t
$\nabla_{\dot{\gamma}_t} A_t : [0, 1] \to \mathrm{End}(T_xM)$ is defined as:

$$\nabla_{\dot{\gamma}_t} A_t(X_t) = (dA_t)_{X_t}(\dot{\gamma}_t) + A_t(\nabla_{\dot{\gamma}} X_t)$$

Let $\mathfrak{t}_x^{(1)} = T_{\mathrm{id}}(G_x^h) \subset B_x(T_xM)$, and for $k \geq 1$: $\mathfrak{t}_x^{(k+1)} = T(\mathfrak{t}_x^{(k)})$.
Then

$$\mathfrak{g}_x^h = \cup_{k=1}^{\infty} \mathfrak{t}_x^{(k)}$$

is called the holonomy algebra of the homogeneous connection.

The curvature: $R(U, V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z$.
Then for $u, v \in T_xM$: $R_{u,v} \in \mathfrak{t}_x^{(2)}$.

Theorem. *(Barthel, 1963)*
For an arbitrary $u_1, u_2, \dots \in T_xM$ *let us construct the mappings*

$$h^{(k)}_{u_1,\dots,u_k} : T_xM \to T_xM$$

by induction:
a) $h^{(2)}_{u_1,u_2} = R(u_1, u_2)$
b) $k > 2$. *Consider* $X_1(t), \dots, X_{k-1}(t)$ *parallel vector fields along* $\gamma(t)$ *with* $X_1(0) = u_1, \dots, X_{k-1}(t) = u_{k-1}$. *Let*

$$h^{(k)}_{u_1,\dots,u_k} := \nabla_{\dot\gamma} h^{(k-1)}_{X_1(t),\dots,X_{k-1}(t)}|_{t=0}$$

Then the vector space $\mathfrak{r}_x$ *spanned by all mappings* $h^{(k)}_{u_1,\dots,u_k}$ *gives a Lie subalgebra of the holonomy algebra* $\mathfrak{g}^h_x$.

We define the notion of hv-**holonomy groups** of the linear Finsler pair connection (∇^F, H):
Consider a loop φ at $x \in M$ and its horizontal lift ψ starting from $z \in TM$: $\pi \circ \psi = \varphi, \quad \psi(0) = z$. ψ is not necessarily a loop, but the endpoints are in the same fibre. Join $\psi(1)$ and $\psi(0)$ with a vertical straight line τ in T_xM. The parallel translations of ∇^F along composite loops $\psi * \tau$ generate the *hv-holonomy group* G_z^{hv} at $z \in TM$.

$$G_z^{hv} = \{P_\tau^F \circ P_\psi^F \,|\, \forall \tau, \psi \text{ above}\},$$

$G_z^{hv} \subset G_z^F$, and is a linear group $\subset \mathrm{Gl}(n)$.

Note that Diaz and Grangier (1976) gave a similar notion of holonomy group G_z^m, called as **mixed holonomy group**. He used the Cartan connection and there the second part of parallel translation was substituted by the canonical isomorphism $\alpha_{z,\bar{z}}\colon \{z\} \times T_xM \to \{\bar{z}\} \times T_xM$, which does not depend on ∇^F. Then $G_z^m \not\subset G_z^F$, only if (∇^F, H) is v-trivial. This would be the case for Berwald and Rund's Finsler-pair connections.

Theorem. *The reducibility of the mixed holonomy group implies that the Finsler space is Riemannian and de Rham decomposition arises.*

Special Finsler spaces:

Minkowski space: $L(x, y)$ does not depend on x

Theorem. *(Heil, 1966) A Finsler space is locally Minkowski space if and only if its holonomy group* G_x^h *is trivial.*

Berwald spaces $\Longrightarrow$ Szabó's lecture

Landsberg space

Definition. *Let* (τ_M, g) *be a Finslerian vector bundle,* H *a homogeneous connection and* g *a Riemannian metric in the Finsler vector bundle* $\pi^*(\tau_M)$.

(τ_M, g, H) *is called* **a Landsbergian vector bundle** *if the Berwaldian Finsler pair connection* (∇^B, H) *is h-metrical.*

Applying the construction of the Berwaldian connection H^B the assumption can be expressed as

$$dL^* \circ s \circ dH^v = 0 \qquad v \in TM \tag{3}$$

Using the covariant derivation of H^B it is equivalent to $\nabla_U g = 0$ for any horizontal $U \in \mathfrak{X}^H(TM)$.

We have the classical notion of Landsberg space if $H = N$, where N is the Barthel homogeneous connection. A series of "iff" conditions is known for Landsberg spaces.

There are a lot of interesting results concerning Landsberg spaces due to S. Dragomir,Z. Kovacs,H. Yasuda. Specially, Landsberg spaces are characterized by the condition that the indicatrix I_p at any point $p \in M$ is a totally geodesic submanifold of the total space ITM of the indicatrix bundle. T. Aikou proved that the Landsberg property is equivalent to that the tangent fibres are totally geodesic submanifolds of TM with respect to the Sasaki metric of TM.

[Aikou, T.: *Some remarks on the geometry of tangent bundle of Finsler manifolds.* Tensor, N. S. **52** (1993), 234-242.]

Consider now the parallel translation with respect to H. It exists for entire curve $\varphi\colon I \to M$. Therefore the parallel translation $P_\varphi\colon T_pM \to T_qM$ is a homogeneous bijective map if φ joins $p \in M$ and $q \in M$. Secondly, a fibre T_pM at a point can be regarded as a Riemannian space by $g(p, z)$ fixing the point p.

Theorem. *(Ichijyo, 1983, Kozma, 1995)*
(τ_M, g, H) *is a Landsbergian vector bundle if and only if the parallel translation of the homogeneous connection is an isometry between the fibres as Riemannian spaces for any curve.*

Theorem. *(Kozma, 1997) The holonomy group of a Landsberg manifold is a compact Lie group.*

Proof. (Sketch) It follows that the holonomy group is a closed subgroup of the isometry group of the fibre considered as Riemannian space. On the other hand the indicatrix remains invariant when applied the holonomies. Take the restriction of the holonomies on the indicatrix. Now the indicatrix at a point x is a compact Riemannian space, therefore its isometry group is a compact Lie group. Thus the holonomy group is a closed subgroup of the compact Lie isometry group, consequently itself is a compact Lie group, too.

□