REMARKS ON THE APPLICATIONS OF FINSLER GEOMETRY TO SPACE-TIME

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Contents:

1. Some fundamental concepts of Finsler Geometry. A relationship with physical principles

The so-called Finsler Geometry which has been originated by Paul Finsler in 1918, it is not so well known yet even at the present time.

Finsler geometry has started with Finsler's famous dissertation under the supervision of C. Caratheodory who intended to geometrize the calculus of variations. However according to Matsumoto the creator of this geometry is L. Berwald in 1925. The name "Finsler Geometry" was first given by J. Taylor in 1927.

In 1854 B. Riemann introduced the so-called Riemann metric

$$ds^2 = g_{ij}\, dx^i dx^j \tag{1}$$

Before arriving at Riamannian metrics he is concerned with the concept of generalized metric

$$ds = F(x^1, x^2, \ldots, x^n, dx^1, \ldots, dx^n) \tag{2}$$

which gives the distance between two points x and $x + dx$.

He discusses the conditions which should be satisfied by the function $F(x, dx)$. Conditions given by him are as follows:

(F1) $F(x, y) > 0$ for any $y = dx$

(F2) $F(x, py) = pF(x, y)$ for any $p > 0$

(F3) $F(x, -y) = F(x, y)$

The condition (F1) expresses that the distance between two points should be positive.

The homogeneity condition (F2) is also natural. If the quantities y^i are multiplied by p then the value of the metric function $F(x, y)$ (distance) should be done by p. The condition (F3) expresses the symmetry of distance between two points. However (F3) is enough restricted. We shall see in the following that (F3) is not necessary (e.g. in physical applications). Also as pointed out by S.S.Shern "Finsler geometry is just Riemannian geometry without a quadratic restriction" (Notes of AMS, Sept. 1996), which is given by rel. (1).

The length s of a curve C: $x^i(t)$, $a \le t \le b$ in a manifold of consideration is given by

$$s = \int_a^b F\left(x(t), y(t)\right) dt \qquad y(t) = \frac{dx}{dt}$$

If C is to be written by another parameter $\tau = \tau(t)$, $c \leq \tau \leq d$ the length of C:

$$s = \int_c^d F\left(x, \frac{dx}{d\tau}\right) d\tau$$

The integral of the length is independent of the parameter if and only if the condition (F2) is valid.

$$F(x, py) = pF(x, y) \qquad p = \frac{d\tau}{dt} > 0$$

Remark 1. The anisotropic character of a Finsler space is expressed in better way by the concept of indicatrix x: fixed, $F(x, y) = 1$.

Remark 2. If the condition (F2) is not valid another class of metric spaces different from Finsler spaces can be introduced, for example Lagrange spaces.

By Euler's theorem of homogeneous functions we can get for the function F homogeneous degree one, the relations.

$$F(x, y) = \frac{\partial F(x, y)}{\partial y^i} y^i \qquad \frac{\partial^2 F(x, y)}{\partial y^i \partial y^j} = 0$$

from which we derive

$$F^2(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} y^i y^j$$

Using an analogous to Riemannian geometry for the norm of a vector we get

$$\frac{1}{2} \frac{\partial^2 F(x, y)}{\partial y^i \partial y^j} y^i y^j := f_{ij}(x, y), \qquad \det(f_{ij}) \neq 0 \tag{3}$$

f_{ij} plays the role of metric tensor or a potential in the Finslerian general relativity (covariant tensor of order two) which come from the function F, so we can call F *generator function or metric function.*

When the metric tensor $f_{ij}(x, \omega)$ depends on an intrinsic variable $\omega = y, z, \xi$ (where y = vector, z = scalar, ξ = spinor) the form of Finsler space changes. In the case that the anisotropic metric tensor f_{ij} depends on spinors a theory of anisotropic space-time in relation with spinors was developed by P.Stavrinos and S.Vacaru [8].

If we consider a Finsler-type space-time the anisotropic metric tensor $f_{ij}(x, y)$ depends on, even locally, the directional variables and thus there are two possibilities in defining the causality (H.Ishicawa, J.Math. Phys. 1981).

1. V^i will be called *null* if $f_{ij}(x,y)V^iV^j = 0$ with respect to the direction y.

2. V^i will be called *null* if $f_{ij}(x,V)V^iV^j = 0$

In Cartan's sense (x,y) represents the element of support. Also a tensor $C_{ijk} = \dfrac{1}{2}\dfrac{\partial f_{ij}}{\partial y^k}$ (was introduced by Cartan) is fundamental significant in Finsler Geometry.

$$C_{ijk} = 0 \Longrightarrow \text{Riemannian geometry} \tag{4}$$

The Christoffel symbols in a Finsler space are constructed by $f_{ij}(x,y)$.

$$\gamma_{ijk}(x,y) = \frac{1}{2}\left(\frac{\partial f_{kj}(x,y)}{\partial x^i} + \frac{\partial f_{ik}(x,y)}{\partial x^j} - \frac{\partial f_{ij}(x,y)}{\partial x^k}\right) \tag{5}$$

are given in same way with of Riemannian space where Γ^i_{jk} are constructed by a Riemannian metric $a_{ij}(x)$.

The quantities $\gamma_{ijk}(x,y)$ are called *Finslerian Christoffel symbols.*

Osculating Riemannian metric.

$$a_{ij}(x) = f_{ij}\left(x, y(x)\right) \tag{6}$$

We suppose that $y(x)$ is vector field defined over a region U of a Finsler space F_n. Then a *Riemannian metric* $a_{ij}(x)$ is defined over U by means of (6).

The region U is called *Osculating Riemannian manifold.*

Christoffel symbols are written

$$a_{ijk} = \frac{1}{2}\left(\frac{\partial a_{kj}(x,y)}{\partial x^i} + \frac{\partial a_{ik}(x,y)}{\partial x^j} - \frac{\partial a_{ij}(x,y)}{\partial x^k}\right) \tag{7}$$

¿From a physical point of view the metric given by (6) and the Christoffel symbols (7) play a fundamental role in some problems of Finsler relativity e.g. for a static gravitational field (Asanov's monograph 1985 [2]).

¿From the variational problem of the calculus of variation

$$\delta \int ds = 0 \qquad ds = F(x, \dot{x})$$

we get the Euler-Lagrange equations in Finsler geometry

$$\frac{d}{ds}\left(\frac{\partial F(x,\dot{x})}{\partial \dot{x}}\right) - \frac{\partial F(x,\dot{x})}{\partial x} = 0$$

where $\dot{x} = \dfrac{dx}{ds}$

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k = 0 \tag{8}$$

The equation (8) yields the *equation of geodesics of a Finsler space.*

In the case of special type of a Finsler space so-called Randers space the relation (8) provides us that the motion of a charged particle in a such Finsler space is a *geodesic* in contrast with the movement of a charged particle in a Riemannian space-time (gravitational field) in which the electromagnetic field is present.

Berwald is the first who introduced the concept of connection in Finsler space (1926).

He introduces the functions

$$2G^i(x,y) = \gamma^i_{jk}(x,y)y^jy^k \tag{9}$$

differentiates it twice

$$G^i_j = \frac{\partial G^i}{\partial y^j} \qquad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k} \tag{10}$$

$G^i_{jk}(x,y)$ are symmetric in subscripts.

The homogeneity condition (F2) allows us to write the differential equation of a geodesic curve (8) in the form

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{11}$$

If we consider the notion of parallel displacement of a vector field V^i along the curve $C : x^i = x^i(s)$ according to Berwald it is defined by the differential equations

$$\frac{dV^i}{ds} + G^i_{jk}(x,y)V^j\frac{dx^k}{ds} = 0 \tag{12}$$

where $y^i = y^i(s)$ is a *direction field a priori given along* C. So rel. (12) is called "parallel displacement" of V^i along C with respect to $y = y(s)$.

By differentiation of $G^i_{jk}(x,y)$ we get

$$\frac{\partial G^i_{jk}(x,y)}{\partial y^l} = G^i_{jkl} \qquad \text{(curvature tensor)}$$

In the case that

$$G^i_{jk}(x) \Rightarrow G^i_{jkl} = 0$$

such a type of space is called *Berwald space*. Namely Berwald connections depends only on the position x. Spaces with $G^i_{jkl} = 0$ called by Berwald "affinely connected spaces" (1925). In these spaces is valid $G_{ijk|l} = 0$ where " | " means Cartan covariant derivative.

Berwald connection are useful to use in Randers (Finsler) spaces [3].

A type of Finsler space so-called Randers space has been studied and developed by a physical point of view by Kilmister, Horvath, Ingarden and G.Asanov [2]. The metric in a Randers space is given by

$$F(x, y) = \sqrt{g_{ij}(x)y^i y^j} + kb_i y^i \tag{13}$$

$$k = \begin{cases} const. \\ \frac{q}{mc^2} \end{cases}$$

The first part of this metric is a pseudo-Riemannian one. In the third part of my presentation we shall deal with a type of a Randers space which is convenient for the study of anisotropic model of space-time.

An important class of generalized non-Riemannian metric spaces have been studied and developed for mathematical and physical purposes by Einstein, Eddington, Synge, H.Weyl, Eisenhart, Veblen and others. These are the *spaces of paths*. In these spaces the metric tensor is not given as in Riemannian geometry. Riemannian Christoffel symbols are substituted by another set of functions $G^\alpha_{\beta\gamma}(x)$.

In order to construct a geometry from these symbols you need only a transformation law

$$G^\gamma_{\alpha\beta}\frac{\partial \bar{x}^\sigma}{\partial x^\gamma} = \overline{G}^\sigma_{\nu\varepsilon}\frac{\partial \bar{x}^\nu}{\partial x^\alpha}\frac{\partial \bar{x}^\varepsilon}{\partial x^\beta} + \frac{\partial^2 \bar{x}^\sigma}{\partial x^\alpha \partial x^\beta} \tag{14}$$

Equations of geodesics

$$\dot{V}^\beta + \overline{G}^\beta_{\gamma\sigma}V^\gamma \dot{x}^\sigma = 0 \tag{15}$$

V^β : contravariant vector

¿From the rel. (14) a more general covariant differentiation can be introduced with respect to $G^\alpha_{\beta\gamma}$.

$$\overline{V}^\alpha_{;l} = \frac{\partial \overline{V}^\alpha}{\partial \bar{x}^\rho} + V^\nu \overline{G}^\alpha_{\nu\rho} \tag{16}$$

If we choose an arbitrary (fixed) path $C : x^a = x^a(t)$ on a manifold, then the solutions $V^\beta(t)$ of (15) corresponding to certain initial conditions, that is to a vector in the manifold given at a point of C. By the relation (15) V^β are said to be parallel with respect to the displacement defined by $G^\alpha_{\beta\gamma}$.

In the framework of a more general type of given functions $H^i(x,y)$, as was suggested by Douglas *the general space of paths* is

$$\frac{d^2x^i}{ds^2} + H^i_{jk}(x,y)y^jy^k = 0 \tag{17}$$

where $H^i_{jk}(x,y) = \dfrac{\partial^2 H^i}{\partial y^j \partial y^k}$ and $y^i = \dfrac{dx^i}{dt}$.

If there is a metric function $F(x,y)$ in the space and hence a metric tensor $f_{ij}(x,y)$ the functions $H^i(x,y)$ are equivalent with $G^i(x,y)$ given by Berwald (which are connected with $\gamma^i_{jk}(x,y)$ by (9)). In this case the set of equations (17) with $t = s$ represent the geodesics of the metric $F(x,y)$.

An application to the general relativity

Finsler geometry is the *geometry of space and motion.*

In our universe we remark that "there is no position without motion". A Finsler space can be considered as a manifold of positions (coordinate systems x^i) and of tangent vectors y^i (velocities) along the curves (world lines of the moving particles) of the background.

The general spaces of paths are closely connected with the principle of equivalence.

In the four dimensional world of space-time the trajectory of a particle *falling freely in a gravitational field* is a certain fixed curve. Its direction at any point depends on the velocity of the particle. The principle of equivalence implies that there is a preferred set of curves in space-time at any point, pick up any direction and there is a unique curve in that direction that will be trajectory of any particle starting with that velocity. *These trajectories are thus the properties of space-time itself.*

This standpoint reveals a profound relation between the principle of equivalence and the space-time of paths in Finsler spaces, namely rel. (11),(12),(15),(17) or rel. (8) of Finsler geodesics are closely related with the equivalence principle.

In addition with a Finsler-Randers type space-time as we shall present in the following, the limits of the equivalence principle of General Relativity can be extended since the presence of the electromagnetic field does not affect the geodesic motion of a charged particle in the space. The electromagnetic field is *intrinsically incorporated* in the geometry of the space.

In Finsler spaces there are different types of connections as *Berwald connections*, *Cartan connections*, *Chern (Rund) connections*, *Hashiguchi* connections. These connections play a crucial role in Finsler geometry and its applications (cf. Bibliography).

Remark

Every type of connections provides us covariant differention. Tensors and curvatures are constructed from these connections. Variational problems (e.g. equations of geodesics) and deviations of geodesics which are important from a mathematical and physical point of view are connected with the tidal forces of the space (which depend on the original connections). We shall mention about them in the part 2.

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2. Riemannian-Finslerian deviations of geodesics and their consequences to physical phenomena

In this part we study the geometrical and physical properties of the equation of geodesic deviation in Riemannian and Finslerian (anisotropic) space-time. We extend the concept of this equation to the of weak fields in a Fisler-Randers space-time.

1 Some types of Riemannian equations of geodesic deviations and their physical principles

The profound role of the geodesic deviation of Riemannian space-time has been recognized in the general relativity for a long time. [1, 2, 3, 4]. The general form of this equation in Riemannian space-time is given by

$$\frac{\delta^2\eta^k}{\delta s^2} + R^k_{lmn}v^l v^m \eta^n = 0 \tag{1}$$

where $\delta/\delta s$ means the usual covariant derivative, R^k_{emn} represents the Riemannian curvature tensor, η^k is the separation vector which separates two nearby free falling material test particles ($\delta \int ds = 0$). This vector represents physically the position between two free-falling particles, v^n is the tangent four-velocity of the particles and the parameter s symbolizes the proper time along the geodesics.

The equation (1) gives the relativistic generalization of the Newtonnian result for the tidal force field. The form of the curvature tensor R^i_{jkl} (tidal field) of (1) expresses *a property of the space*. How does the small distance between corresponding points vary as they move along the geodesics? This is the problem of geodesics deviation and if we can solve it we get a good insight into the nature of the space. When the geodesics are time-like or null this problem is based on observational tidal phenomena.

In the Riemannian space-time the most important forms of the curvature tensor appear to physical phenomena in two cases. When the space is of constant curvature tensor K and when the gravitational field is weak.

In the first case the form of the Riemannian curvature tensor given by (1) is written in the form

$$R_{ijkl} = K\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \tag{2}$$

If we set into correspodence the events on two neighbouring spacelike, timelike or null geodesics C, C' in a space of constant curvature K the equation of their deviation reduces to

$$\frac{d^2}{ds^2}\left(n^i v_i\right) + (\varepsilon)Kn^i v_i = 0 \tag{3}$$

where $\varepsilon = \pm 1$ and v^i represents any vector propagated parallely long to C. In this space the equation of geodesics deviation may be integrated.

The deviations of spacelike and timelike geodesics are given by

$$n^i v_i = \alpha \sin s(\varepsilon K)^{1/2} + \beta \cos s(\varepsilon K)^{1/2} \qquad \text{if } K > 0 \tag{4}$$

$$n^i v_i = \alpha s + \beta \qquad \text{if } K = 0 \tag{5}$$

$$n^i v_i = \alpha \sinh s(-\varepsilon K)^{1/2} + \beta \cosh s(-\varepsilon K)^{1/2} \qquad \text{if } K < 0 \tag{6}$$

Null geodesics are of great importance in relativity because nearly all astronomical information comes to us optically i.e. by photons in space-time. The deviation vector in this case takes the form

$$\eta^2 = \alpha' s^2 + \beta' s + \gamma$$

where α', β', γ are constants. The relation (2) is closely connected with the observable microwave background radiation [6] which we will discuss in the next part (§3) in the framework of an anisotropic Finslerian space-time.

The equation of geodesics deviation (1) can also be set in the form of orthonormal tetrads $\lambda^i_{(\alpha)}$ and to be investigated in relation with the Fermi and optical coordinates which are useful for the study of physical observations [1, 4].

In the second case we consider the gravitational field to be weak. The weakness of the gravitational field is expressed at least in a certain region of space-time to decompose the metric into the flat Minkowski metric plus a small perturbation (this decomposition of the metric is not unique).

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| << 1, \tag{7}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{8}$$

In the solar system for example we have

$$|h_{\mu\nu}| \sim \phi|/c^2 \lesssim GM_\odot/c^2 R_\odot \sim 10^{-6}.$$

However, the field can vary with time as in the case of gravitational waves, therefore there are no restrictions on the motion of test particles.

The Christoffel symbols and the Ricci curvature tensor in linearized form given by

$$\gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu} + h^{\alpha}_{\nu,\mu} - h^{,\alpha}_{\mu\nu}\right) \tag{9}$$

$$H_{\mu\nu} = \frac{1}{2}\left[h^{\lambda}_{\mu,\lambda\nu} - \Box h_{\mu\nu} - h^{\lambda}_{\lambda,\mu\nu} + h^{\lambda}_{\nu,\lambda\mu}\right] \tag{10}$$

$$H = h^{\lambda\sigma}_{\lambda\sigma} - \Box h \qquad h : h^{\lambda}_{\lambda} = n^{\alpha\beta} h_{\alpha\beta} \tag{11}$$

where the symbol $\Box$ means the D' Alembertian of the flat space-time. The perturbations $h_{\mu\nu}$ can be determined in a linearized field theory by the field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ in the terms of the "flat" energy-momentum tensor $T^{\mu\nu}$. This consideration permits us to determine $g_{\mu\nu}$ in the form $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$. $G_{\mu\nu}$ represents here the linear approximation of Einstein tensor. Although $T_{\mu\nu}$ produces a weak gravitational field this does not react back on the source. For example in the case that we consider an incoherent dust ($T^{\mu\nu} = \rho u^\mu u^\nu$, $u^\mu = \frac{dx^i}{ds}$ is the four-velocity of the stream-lines, ρ is the density of the dust) [5]. An interesting problem in the study of the linearized gravitation is that it may predicts magneto-gravitational results by the motion of masses, similar to those of electromagnetism [6]. The weak gravitational field reveals some physical effects in the space which can be created by the appearence of gravitational waves in the form of traveling tidal forces.

In order to detect a gravitational wave, at least two particles are needed. The propagation of a gravitational wave through space-time produces a relative deviation of two nearby geodesics and this deviation can be measured in terms of the vector η^μ. The separation of nearby free-falling masses is governed by equation (1). One can determine certain components of the curvature tensor R_{rlmn}, for example observing the shift of interference fringes with a Michelson interferometer (for more details [3]).

In this case the equation deviation (1) reduces to

$$\frac{d^2\xi^k}{d\tau} = -H^k_{0l0}\xi^l \quad (\tau : \text{proper time}) \qquad (\xi : \text{deviation vector}) \tag{12}$$

for a freely falling geodesic reference frame (Fermi coordinates) of the masses.

The tidal force is therefore $f^k = -mH^k_{0l0}\xi^l$ (m: mass of the particles) and can be used for measuring the components H^k_{0l0} of the Riemann tensor H^k_{ilj}. The equation (7) is useful for the description of the polarization of the wave [7].

2 Deviations of geodesics and weak fields in Finsler-Randers space-time

The concept of geodesic deviation can be studied in a Finsler space in analogy with the Riemannian case [8, 9, 10, 11]. Some physical interpretations have been studied in different forms of Finsler spaces [9], [10], [15].

In a general Finsler space all the geometrical elements depend on the position x as well as the direction $y = dx/dt$ as we have mentioned in the part one. For example considering in a Finsler space a point x and a differentiable curve passing through x it is necessary to consider its tangent vector y at the point x. This vector defines a *direction* in that point. The couple (x, y) consists the so-called *supporting element* of the Finsler space. Analytically, a Finsler space is defined as a triplet (M, V, F) where $M = M^n$ is a differentiable manifold of class C^∞, $V = \bigcup_{x\in M} V_x$ is a cone bundle, V_x is a cone (V_x: cone when $y_1 \in V_x$ then $y_2 = py_1 \in V_x$, $p > 0$) in the tangent space $T_x(M)$, $F : V \setminus \{0\} \to \mathbb{R}_+$ is a positive function $F(x, y)$, $x \in M$, $y \in V_x$, $y \neq 0$ of class C^∞ satisfying the following conditions:

1. F is (1) y-homogeneous,
   $F(x, ky) = kF(x, y)$, $k > 0$

2. F is metrically regular, i.e.,

$$|f_{ij}| := \left| \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right| \neq 0 \tag{13}$$

F is called the generator metric function or the Lagrangian. The tensor $f_{ij}(x, y)$ is called as in Riemannian geometry the metric tensor but it has here a more general meaning since it depends, in general, also on the direction y being by this *anisotropic*. The length of a curve $C : t \in [0, 1] \to \vec{x}(t) \in M$ such that $y(t) = dx/dt = \dot{x}(t) \in V_x$ for all $t \in [0, 1]$ is defined by the integral

$$\begin{aligned} s &= \int_0^1 F(x(t), y(t))dt = \int_C F(x, dx) \\ &= \int_0^1 [f_{ij}(x(t), y(t))y^i(t)y^j(t)]^{1/2} dt \end{aligned} \tag{14}$$

which is independent of the parametrization of C. Instead of (14) we can also write

$$ds = F(x, \dot{x})dt = F(x, dx) = [f_{ij}(x, \dot{x})dx^i dx^j]^{1/2}. \tag{15}$$

In a Finsler space there are three types of curvatures R^i_{jkl}, P^i_{jkl}, $S^\alpha_{\beta\gamma\delta}$, for more details someone can see in the monographs [8], [12], [18].

In the framework of Finler geometry the equation of geodesic deviation is given in the following forms:

$$\frac{\delta^2 z^i}{\delta u^2} + K^i_{jhk}(x, y)y^j y^h z^k = 0 \qquad \text{(Rund type)} \tag{16}$$

$$\frac{\delta^2 z^j}{\delta u^2} + H^j_k(x, y)z^k = 0 \qquad \text{(Berwald type)} \tag{17}$$

where the "deviation tensor" is defined by

$$H^j_k(x, y) = K^j_{ihk} y^i y^h$$

An extension of the concept of geodesic deviation to the vertical geodesics in the "Tangent Riemannian" space-time with metric

$$ds_x^2 = g_{ij}(x, y)dy^i dy^j \qquad x : const.$$

was given by P. Stavrinos (1992) [14] and P. Stavrinos-H. Kawaguchi (1993) [11].

$$\frac{D^2\xi^i}{ds^2} + S^i_{jkl}\xi^k n^m \xi^l F^{-2} = 0$$

where S^i_{jkl} is the curvature tensor of Cartan (similar to Riemannian curvature tensor). S^i_{jkl} is being associated by Cartan connection coefficients $S^i_{jk}(y)$.

Physical applications of this equation can be studied for an isospin space.

Generalized Finslerian deviation equations can be investigated in the low-velocity approximation approach for static and spherically-symmetric case of the Finslerian gravitational field. The corrections of Finslerian type with respect to motion (velocity) are of experimental significance for all related observations (tidal forces behaviour or the deviation of trajectories of nearby spacecrafts).

Explicit formula for the main term of the geodesic deviation equation was given in [15].

In the framework of the fibered Finsler gauge approach the deviation equation has been studied by Asanov and Stavrinos in [10]. The form of equation is seperated in the horizontal part and in the vertical part. The most important part is the vertical by a physical point of view since it contains a Yang-Mills-type gauge tensor $\varepsilon^{\beta}_{\alpha ik}$.

¿From the horizontal and vertical parts of the geodesic deviation we get the information:

How the geometry of a fibre influences the behaviour of the background geodesics.

which is equivalent to the statement:

Appearance of additional terms plays the role of additional forces.

In the framework of Lagrange spaces some results of E.D.G. were given by Balan and Stavrinos (1996-97) [17] and an extension in Higher-Order Lagrange spaces by Miron-Balan-Stavrinos-Tsagas (1996) [16].

In a Finslerian space-time the gravitational field $f_{\mu\nu}(x,v)$, $v = dx/dt$ and the Euler-Lagrange equations of curves of the space are related with the relativistic principle equivalence proposed by Einstein giving the same extremal curves (geodesic) with initial conditions (x_0, v_0). That means the line element of the space with a starting point x_0 and v_0 the initial velocity or direction of the path of the particle. An analogous situation can be interpreted to the Newtonian principle of equivalence and the Finsler space.

The behavior of particles in a gravitational and electromagnetic field is expected to indicate that the real geometry in the direction of the unification is the Finsler geometry.

In a Finsler space the metric function $F(x, y)$ can be considered as a potential function since the metric tensor (gravitational potential) is produced by this function. In a Randers space the potential function is given by

$$F(x, y) = \sqrt{\gamma_{ij}(x) y^i y^j} + kA_i(x) y^i \tag{18}$$

where γ_{ij} represents the Riemannian metric, $y^i = dx^i/d\lambda$, λ : parameter along a curve, k : constant. A_i represents the electromagnetic potential. The metric tensor of a Randers space is given by virtue of (18) in the form

$$f_{ij} = \gamma_{ij} + \frac{2k}{\sigma} y^s \gamma_{s(i} A_{j)} + k^2 A_i A_j + \frac{k}{\sigma} y^l A_l h_{ij} \tag{19}$$

where $\sigma = \sqrt{\gamma_{ij} y^i y^j}$, $h_{ij} = \gamma_{ij} - \sigma^{-2} \gamma_{si} \gamma_{jl} y^s y^l$ and in the lower indices (i, j) symbolizes the symmetric summation.

We observe that whenever an electromagnetic field exists in a region of spacetime the geometry becomes Finslerian and the isotropy breaks [13]. The explicit form of Randers connection coefficients are given by

$$F^l_{ij} = \gamma^l_{ij} + E^l_{ij} \tag{20}$$

where γ^l_{ij} are the Riemannian Christoffel symbols and E^l_{ij} are quantities which depend on F^j_i and the metric γ_{ij}

We note from (20) that the electromagnetic field enters in the connection coefficients of the space.

The geodesics of the Randers space are given by

$$\frac{dy^m}{d\lambda} + F^m_{ij}(x, y) y^i y^j = 0 \tag{21}$$

Inserting the relation (20) to (21) we get the well known Lorentz equation

$$\frac{d^2 x^m}{d\lambda^2} + \gamma^m_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} + kF^m_j \frac{dx^j}{d\lambda} = 0 \tag{22}$$

where k is a constant.

The relation (21) represents the equation of motion of a charged particle in a gravitational and electromagnetic field results naturally as the geodesic of Finsler space-time.

The curvature tensor of Randers space is produced by the connection coefficients by analogy of a Berwald space [12].

$$\overline{H}^{i}_{hjk}(x,y) = R^{i}_{hjk} + E^{i}_{hjk} \tag{23}$$

where R^i_{hjk} is the Riemannian curvature and E^i_{hjk} represents the "electromagnetic curvature".

Remark 1 In the general relativity the force of gravity is built into the structure of space-time and exhibits itself in the curvature of space-time. We recognize as forces only the effects of mechanical stresses or electromagnetic fields.

In Finslerian relativity the space-time is constructed with an anisotropic metric (13) $f_{ij}(x,y)$, $y = dx/dt \equiv v(t)$, for example in a Finsler-Randers space with the metric of the form (18). In this case the metric of space-time is constructed with the gravitational field (curvature $R_{ijkl}(x,v) \neq 0$) as well as the electromagnetic curvature $H_{ijkl} \neq 0$ which was mentioned in (23).

Remark 2 In a Finsler-Randers space every particle moving along a geodesic of the space satisfies the Lorentz equation. This is identified with the Lorentz equation of a particle moving in a curve of the gravitational and electromagnetic field of the Riemannian space-time (not geodesic).

There are some cases that Randers space consists of a good application of physics of the Finsler spaces. Under a linearized approach of the gravitational field the Randers proper time interval can be written in the form of a first approximation of the Riemannian metric γ_{ij}.

$$d\tau = F^{-1}(x,v)\left(\sqrt{\left(n_{ij} + \varepsilon^{(1)}_{ij}\right) v^i v^j + kA_i v^i}\right) dt \tag{24}$$

where $v^i = dx^i/dt$ represents the four velocity of the particle with respect to the proper time τ, $|\varepsilon^{(1)}_{ij}| \ll 1$, $\varepsilon^{(1)}_{ij}$ represents the small corrections to the flat space-time metric n_{ij} and k to be a constant. The linearized form of the metric tensor by (19) is given now by

$$f_{ij} = \left(\eta_{ij} + \varepsilon^{(1)}_{ij}\right) + \frac{2k}{\sigma'} v^s \eta_{s(i}A_{j)} + k^2 A_i A_j + \frac{k}{\sigma'} v^l A_l \theta_{ij} \tag{25}$$

where $\sigma' = \sqrt{\eta_{ij}v^i v^j}$, $\theta_{ij} = \eta_{ij} - \sigma'^{-2}\eta_{si}\eta_{jl}v^s v^l$.

Considering in (25) the case where $v^s = (1, 0, 0, 0)$ we get for the Finslerian potential f_{00} of the Randers space

$$f_{00} = -1 + \varepsilon_{00}^{(1)} + k^2\phi^2 + k\phi \tag{26}$$

with $\phi = A_{00}$. The Finslerian potential f_{00} takes the value -1 for the values of φ:

$$\varphi_{1,2} = \left[-1 \pm (1 - 4\varepsilon_{00}^{(1)})^{1/2}\right](2k)^{-1} \tag{27}$$

This is useful to derive the Riemannian or Newtonian limit from the equation of motion of the form

$$\dot{v}^l + F_{00}^l v^0 v^0 = 0$$
$$\text{or} \qquad \dot{v}^l + F_{00}^l = 0$$

where the full interpretation F_{lm}^k is given by (20).

In this case the Christoffel symbols and the curvature tensor of the Randers space will take the form is given (28), (29). So we get from (20), (23) the relations

$$\bar{h}_{lj}^i = \varepsilon_{lj}^i + h_{lj}^i \tag{28}$$

$$\bar{h}_{ljk}^i = \varepsilon_{ljk}^i + h_{ljk}^i \tag{29}$$

where ε_{jk}^i, ε_{jkl}^i are the linearized Riemannian Christoffel symbols and the curvature tensor. The terms h_{jk}^i, h_{jkl}^i are given explicitly in [13].

Using the linearized connection coefficients by (28) we take the Lorentz equation of the weak field in a Randers space

$$\frac{dv^m}{d\tau} + \varepsilon_{ij}^m v^i v^j + kF_j^m v^j = 0 \tag{30}$$

The deviation of geodesics in Randers space is given by

$$\frac{\delta^2\xi^i}{\delta\lambda^2} + \overline{H}_{jhk}^i(x, v)\xi^j v^h v^k = 0 \tag{31}$$

$$\text{where} \quad \frac{\delta x^i}{\delta\lambda} = \xi_{|h}^i V^h, \qquad \xi_{|h}^i = \frac{\partial\xi^i}{\partial x^h} + L_{hk}^i(x, v)\xi^k$$

L^i_{hk} are the Cartan connection coefficients , ξ^i represents the deviation vector and v^k the tangent vectors of a geodesic surface included in the Randers space-time. We note from (31) that the deviation equation has two terms : one which corresponds to the gravitational deviation that we would observe if there was no electromagnetic field and which is associated with the R^i_{hjk} part of the curvature tensor. The other one corresponds to a mixed geometrical and electromagnetic deviation and is associated with the E^i_{jhk} part of the $\overline{H}^i_{hjk}$ tensor. It would be interesting to study the second term of deviation trying to connect it with the electric force that two freely falling charged particles would exert each other. In such a case this force would result naturally as a geometrical effect in a Finsler space-time and would not be necessary to impose it additionally as it seems in a Riemannian space.

When $R^i_{jhk} = 0$, from (31) we infer that the first term of the Randers metric corresponds to a Minkowski metric and the Finsler (Randers) space becomes v-locally Minkowski.

$$F(x,v) = \sqrt{\eta_{\mu\nu}v^\mu v^\nu} + kA_i(x)v^i \tag{32}$$

The only force that influences the two particles is due to the presence of the electromagnetic field. The deviation equation takes the form [13]

$$\frac{\delta^2 z^i}{\delta u^2} + E^i_{jkl} z^j v^k v^l = 0 \tag{33}$$

In this case the geometrical properties of the field are characterized by a homogeneous and anisotropic space. The metrical fundamental tensor depends only on the velocities which produce the anisotropic properties of the *curved Finsler space-time*. Consequently there exists a frame of reference in which $F^i_{jk} = 0$. This result is convenient if one uses the *harmonic* coordinate condition for studying a linearized field theory in a Finsler space. Under these circumstances the geodesic coordinates can be introduced for the particles which move along these geodesics. From (33) we infer that the propagation of an electromagnetic wave produces a relative deviation of the two nearby geodesics (electrical potential lines). In a similar way the propagation of a gravitational wave through the space-time produces a relative deviation of the mass-particles moving to the nearby geodesics.

In order to study the weak field of a Randers space related to the deviation of the two charged particles it is necessary to take into account the relations (29), (31), (33). This is reasonable since in order to detect a gravitational wave at least two particles are needed. So the deviation of geodesics of the weak Randers space is written in the form

$$\frac{D^2\xi^i}{d\tau^2} + \left(\varepsilon^i_{ljm} + h^i_{ljm}\right)\xi^j \frac{dx^l}{d\tau}\frac{dx^m}{d\tau} = 0 \tag{34}$$

The relation (34) in a first approximation of h_{ij} takes the form

$$\frac{\partial^2\xi^i}{\partial t^2} = -\left(\frac{1}{2}\frac{\partial^2\varepsilon^i_j}{\partial t^2} + 2F^i_0 F^0_j + u_j\frac{\partial F^i_0}{\partial t}\right)\xi^j$$

The equation (35) coincides with the corresponding equation for a weak field of the Riemannian space-time [3]. The difference between them is that in the Riemannian case the electromagnetic field has been introduced ad hoc. In the equation (31) of the Randers space the electromagnetic field is incorporated in the geometry and the two charged particles are moved in geodesics of the Finsler space, and their relative acceleration is governed by the curvature of the gravitation and electromagnetic field which is produced by the energy-momentum tensor.

3 Discussion

We studied the geometrical and physical properties of the geodesic deviation in Riemannian and Finslerian space-times. We also investigated the weak field in relation with the deviation of geodesics of a Finsler-Randers space and we revealed some profound properties of the space by virtue of equations (22), (28), (35). Gravitational waves can be produced by perturbations of the weak curvatures in Finslerian space-time. Possible relationships between gravitational waves and the deviation equation can be established to this space-time analogously with the Riemannian case.

The electromagnetic field is intrisically incorporated in this geometry. Under this circumstances we extended the concept of the geodesic deviation in the Finsler-Randers space-time.

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3. On the anisotropic structure of Finslerian space-time

1 Introduction

In the last years the observational results are useful to confirm the possibility in the anisotropic expansion of the universe [11].

Anisotropic *direction-dependent* expansion may be presented if the underlying geometry of the universe is anisotropic (in which case the anisotropic Roberston-Walker metric is no longer valid or if anisotropic non gravitational forces are present such as a large-scale magnetic field, Thorn 1967).

It is therefore necessary to take seriously the possibility that the Universe is anisotropic and to investigate what effect *anisotropic expansion* will produce on the *angular distribution of the background radiation.*

On the other hand S.Weinberg notices in his book entitled *"The first three minutes"*:

"It is possible the whole universe we can see looking backwards the elapsed time can be an isotropic and homogeneous clot within a non-homogeneous and anisotropic universe".

In a previous paper [10] we studied the Finslerian structure of space-time caused by the observed anisotropy of the microwave cosmic background radiation. Finslerian geometrical models which can correspond to anisotropic structures of regions of space-time can be introduced.

It is known that the biggest part of this anisotropy can be explained if we use Robertson-Walker metric and take into account the movement of our galaxy with respect to distant galaxies of the universe [9]. However a small anisotropy is expected, due to anisotropic distribution of galaxies in space [8].

¿From the above mentioned results it is reasonable to seek for a Lagrangian with respect this anisotropy. As such we choose:

$$\mathcal{L} = \sqrt{a_{ij}y^i y^j} + \phi(x)\hat{k}_a y^a \tag{1}$$

where $y^a = \dfrac{dx^a}{dt}$.

The vector $\hat{k}_a$ *expresses the observed anisotropy* of the microwave background radiation.

In this work we study the Finslerian space-time based on (1). In this framework of our study we use the Berwald-type connection as well as Cartan connection for the s-anisotropic curvature S^i_{jkl} of the space. Also we derive the Berwald curvature tensor. The scalar Riemannian curvature is defined for two directions one of them contains the axis of anisotropy. Also we can examine some cases in which the scalar curvatures is constant.

In paragraph 4 we study the consequences that are caused by a generalized D' Alembertian.

2 Preliminaries

Here we adopt the connection introduced by Berwald as we mentioned previously. In this case the connection coefficients are defined by G^i_{jk},

$$G^i_{jk} = \frac{\partial G^i_j}{\partial y^k} \tag{2}$$

$$G^i_j = \frac{\partial G^i}{\partial y^j} \tag{3}$$

where G^i are defined by the relation:

$$G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k \tag{4}$$

where γ_{ijk} are the Christoffel symbols of Finsler space defined as

$$\gamma_{ijk}(x,y) = \frac{1}{2}\left(\frac{\partial f_{kj}(x,y)}{\partial x^i} + \frac{\partial f_{ik}(x,y)}{\partial x^j} - \frac{\partial f_{ij}(x,y)}{\partial x^k}\right)$$

¿From Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial y^a}\right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0, \qquad y^a = \frac{dx^a}{d\lambda} \tag{5}$$

we derive the equations of geodesics

$$\frac{d^2 x^i}{d\lambda^2} + 2G^i(x,y) = 0 \tag{6}$$

where λ is a special affine parameter (for example the proper time).

The Berwald curvature tensor is written by

$$H^i_{hjk} = \frac{\partial G^i_{hj}}{\partial x^k} - \frac{\partial G^i_{hk}}{\partial x^j} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{rj} + G^r_j \frac{\partial G^i_{rh}}{\partial y^k} - G^r_k \frac{\partial G^i_{rh}}{\partial y^j} \tag{7}$$

3 The Geometrical Structure of the anisotropic model

In the following, the lowering and raising of the indices of the objects $\hat{k}_a, y^a$ and all related Riemannian tensors will be performed with the metric a_{ij}. For the related Finslerian tensors we shall use the Finsler metric f_{ij}.

The Lagrangian which gives the equation of geodesics in the case of (pseudo)-Riemannian space-time is given by:

$$L = \sqrt{a_{ij} y^i y^j}, \quad y^i = \frac{dx^i}{ds} \tag{8}$$

or, equivalently, we may write for the line element:

$$ds_R = \sqrt{a_{ij} dx^i dx^j} \tag{9}$$

where a_{ij} is the Riemannian metric with signature $(-, +, +, +)$. Because of the observed anisotropy, we must insert an additional term to the Riemannian line element (9). This term must fulfill the following requirements:

(a) It must give absolute maximum contribution for direction of movement parallel to the anisotropy axis.

(b) It must give zero contribution for movement in direction perpendicular to the anisotropy axis, i.e. the new line element must coincide with the Riemannian one for direction vertical to the anisotropy axis.

(c) It must not be symmetric with respect to replacement $y^a \to -y^a$. This requirement is necessary in order to express the anisotropy of dipole type of the Microwave Background Radiation (MBR). We need to have maximum (positive) contribution for direction that coincides with the direction of the anisotropy axis, and minimum (negative) contribution for the opposite direction.

We see that a term which satisfies the above conditions is $k_a(x)y^a$, where $k_a(x)$ expresses this anisotropy axis. For constant direction of $k_a(x)$ we may consider $k_a(x) = \phi(x)\hat{k}_a$, where $\hat{k}_a$ is the unit vector in the direction $k_a(x)$. Then $\phi(x)$ plays the role of "length" of the vector $k_a(x)$, $\phi(x) \in \mathbb{R}$. Hence, we have the Lagrangian

$$\mathfrak{L} = \sqrt{a_{ij}y^i y^j} + \phi(x)\hat{k}_a y^a \tag{10}$$

¿From (10) we define the Finsler metric function $F(x,y) = \mathfrak{L}$. Setting $y^a = dx^a$ we have

$$ds_F = \sqrt{a_{ij}dx^i dx^j} + \phi(x)\hat{k}_a dx^a \tag{11}$$

ds_F is the Finslerian line element and ds_R is the Riemannian one. We see that the Finslerian line element is generated by an additional increment to the Riemannian one due to the anisotropy axis. Now

$$ds_F^2 = a_{ij}dx^i dx^j + 2\phi(x)\hat{k}_a dx^a \sqrt{a_{ij}dx^i dx^j} + \phi^2(x)\hat{k}_a dx^a \hat{k}_b dx^b \tag{12}$$

In order for the Finslerian metric to be physically consistent with General Relativity theory, it must have the same signature with the Riemannian metric $(-,+,+,+)$. We have

$$ds_R = c\,d\tau = c\,\gamma dt = \gamma d(ct) = \gamma dx^0 \tag{13}$$

where $\gamma = \sqrt{1-(v/c)^2}$ and v: 3-velocity in Riemannian space-time. ¿From relations (13),(12) we obtain:

$$\begin{aligned} ds_F^2 = {} & a_{00}dx^0dx^0 + 2a_{0\alpha}dx^0dx^\alpha + a_{\alpha\beta}dx^\alpha dx^\beta + 2\phi(x)\hat{k}_0 dx^0 ds_R \\ & + 2\phi(x)\hat{k}_\alpha dx^\alpha ds_R + \phi^2(x)\hat{k}_0\hat{k}_0 dx^0dx^0 + 2\phi^2(x)\hat{k}_0\hat{k}_\alpha dx^0 dx^\alpha \\ & + \phi^2(x)\hat{k}_\alpha dx^\alpha \hat{k}_\beta dx^\beta \end{aligned}$$

or

$$\begin{aligned} ds_F^2 = {} & \left(a_{00} + 2\gamma\phi(x)\hat{k}_0 + \phi^2\hat{k}_0\hat{k}_0\right) dx^0dx^0 + \\ & + \left(a_{\alpha\beta} + \phi^2(x)\hat{k}_\alpha\hat{k}_\beta\right) dx^\alpha dx^\beta + 2\gamma\phi(x)\hat{k}_\alpha dx^\alpha dx^0 \\ & + 2a_{0\alpha}dx^0dx^\alpha + 2\phi^2(x)\hat{k}_0\hat{k}_\alpha dx^0dx^\alpha \end{aligned} \tag{14}$$

where α, $\beta = 1,2,3$. ¿From relation (14) it is evident that we must have

$$(k_0(x))^2 + 2\gamma k_0(x) + a_{00} < 0 \tag{15}$$

$$\delta^{\alpha\beta}\left(a_{\alpha\beta} + k_\alpha(x)k_\beta(x)\right) > 0 \tag{16}$$

for the signature to be preserved, where we have written $\phi(x)\hat{k}_i = k_i(x)$. Relation (15) admits negative values for

$$-\gamma - \sqrt{\gamma^2 - a_{00}} < k_0(x) < -\gamma + \sqrt{\gamma^2 - a_{00}} \tag{17}$$

while from (16) yields:

$$(k_\alpha(x))^2 > -a_{\alpha\alpha} \tag{18}$$

which is true for any $k_a(x)$ since $a_{\alpha\alpha} > 0$.

Then for any physically acceptable vector, its 0 component $k_0(x)$ must lie in the interval (17).

The equation of geodesics is given by:

$$\frac{d^2x^l}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij} y^i y^j + \sigma a^{lm}(\partial_j \phi \hat{k}_m - \partial_m \phi \hat{k}_j) y^j = 0 \tag{19}$$

We observe that in the equation of geodesics we have an additional term, namely $\sigma a^{lm}(\partial_j(\phi\hat{k}_m) - \partial_m(\phi\hat{k}_j))y^j$ which expresses rotation of the anisotropy axis.

Now for the case of electromagnetic waves we must modify relation (19). This is because the world line of an e.m. wave is null. In geometrical optics the direction of propagation of a light ray is determined by the wave vector tangent to the ray. Let $\overset{w}{k}{}^l = dx^l/d\lambda$ be the four-dimensional wave vector, where λ is some parameter varying along the ray. We have:

$$\frac{d\overset{w}{k}{}^l}{d\lambda} + \overset{(a)}{\Gamma}{}^l_{ij} \overset{w}{k}{}^i \overset{w}{k}{}^j + \sigma a^{lm}(\partial_j \phi \hat{k}_m - \partial_m \phi \hat{k}_j) \overset{w}{k}{}^j = 0 \tag{20}$$

One possible explanation of the anisotropy axis could be that it expresses the resultant of the spin densities of the angular momenta of galaxies in a restricted region of space ($k_a(x)$ spacelike). It is known that the mass is anisotropically distributed for regions of space with radius$\leq 10^8$ light years [6]. Then an important kind of anisotropy might result from the ordering of the angular momenta of galaxies. As we move to greater distances (radius$\geq 10^8$ l.y.) the resultant of the spin densities is approximately zero, as it is expected for an isotropic universe.

$$k_a(x) = \sum_i = \overset{(i)}{k}{}_a(x) \approx 0 \tag{21}$$

where $\overset{(i)}{k}_a(x)$ is the spin density tensor of each rotating mass distribution. It may be possible that the restriction (17) expresses the fact that we can not have a largely anisotropic universe. Then the anisotropy vector can not take arbitrary values.

The spin is defined through the spin density tensor [2] from the relation

$$s_{ab} = \frac{\sqrt{-g}}{4\pi}\epsilon_{abc}k^c(x) \tag{22}$$

In the case that $\phi(x)\hat{k}_a$ expresses spin density, the function $\phi(x)$ is related to mass density (angular momenta depends upon angular velocity and mass distribution).

¿From equation (19) we see that for small variation of the resultant of the spin densities vector, the deviation from the Riemannian geodesics is very small, if not negligible.

¿From the equation of geodesics (19) we obtain for movement y^i perpendicular to k^i:

$$\frac{d^2x^a}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij}y^iy^j + \sigma a^{lm}\partial_j\phi\hat{k}_m y^j = 0 \tag{23}$$

¿From (23) it is evident that although the contribution to the ds_R line element is zero for y^i vertical to k^i, the equation of geodesics is different from the Riemannian case. In the case, however, where $\partial_i\phi(x)$is parallel to $\hat{k}_i$, i.e. the increment of anisotropy takes place only along the anisotropy axis, then the equation of geodesics is identical with the geodesics of the Riemannian space-time.

Using the notation $\beta = \hat{k}_a y^a$, $\sigma = \sqrt{a_{ij}y^iy^j}$, we calculate the metric tensor:

$$f_{ij} = \frac{F}{\sigma}a_{ij} + \frac{\phi(x)}{2\sigma}\underset{ij}{\mathcal{S}}\left(y_i\hat{k}_j\right) - \frac{\beta\phi(x)}{\sigma^3}y_iy_j + \phi^2(x)\hat{k}_i\hat{k}_j \tag{24}$$

where $\underset{ij}{\mathcal{S}}$ is an operator and denotes symmetrization of the indices i, j, e.g.

$$\underset{ij}{\mathcal{S}}(A_{ikjl}) = \frac{1}{2}(A_{ikjl} + A_{jkil}).$$

Accordingly we define the antisymmetric operator

$$\underset{ij}{\mathcal{A}}(M_{ikjl}) = \frac{1}{2}(M_{ikjl} - M_{jkil}).$$

The inverse metric is

$$f^{ij} = \frac{\sigma}{F}a^{ij} - \frac{\sigma\phi}{2F}\underset{ij}{\mathcal{S}}\left(y^i\hat{k}^j\right) + \frac{\phi(\beta+m\sigma\phi)}{F^3}y^iy^j \tag{25}$$

as it may be verified by direct calculation, where $m = \hat{k}_a\hat{k}^a = \pm 1$ according weather $\hat{k}_a$ is spacelike or timelike (in order to not loose generality, we do not identify $\hat{k}_a$ as spacelike). The determinant of the metric is

$$f = \det(f_{ij}) = \left(\frac{F}{\sigma}\right)^5 \det(a_{ij}) \tag{26}$$

The finslerian Christoffel symbols of the first kind are given by

$$\gamma_{ijl} = \frac{F}{\sigma}\overset{(a)}{\Gamma}_{ijl} + \Lambda_{ijl} + M_{ijl} \tag{27}$$

where

$$\overset{(a)}{\Gamma}_{ijl} = \frac{1}{2}(\partial_i a_{lj} + \partial_j a_{il} - \partial_l a_{ij}) \tag{28}$$

are the Christoffel symbols corresponding to the metric a_{ij}.

$$\Lambda_{ijl} = \underset{ij\{l\}}{\mathcal{G}}\left[\left(\frac{3\beta\phi}{2\sigma^5}y_iy_j - \frac{\phi}{\sigma^3}\underset{ij}{\mathcal{S}}y_i\hat{k}_j - \frac{\phi\beta}{4\sigma^3}a_{ij}\right)\partial_l a_{ab}y^ay^b\right] \tag{29}$$

and

$$M_{ijl} = \underset{ij\{l\}}{\mathcal{G}}\left[\left(\frac{\beta}{2\sigma}a_{ij} + \frac{1}{\sigma}\underset{ij}{\mathcal{S}}y_i\hat{k}_j - \frac{\beta}{\sigma^3}y_iy_j + 2\phi\hat{k}_i\hat{k}_j\right)\partial_l\phi\right]. \tag{30}$$

The operator $\underset{ij\{l\}}{\mathcal{G}}$ denotes an interchange of the indices in the form this interchange appears in the definition of the Christoffel symbols of a metric, e.g.

$$\underset{ij\{l\}}{\mathcal{G}}A_{ijl} = A_{lji} + A_{ilj} - A_{ijl}$$

$$\underset{ij\{l\}}{\mathcal{G}}\partial_l a_{ij} = 2\overset{(a)}{\Gamma}_{ijl}$$

The Christoffel symbols of the second kind are

$$\gamma^l_{ij} = \overset{(a)}{\Gamma}{}^l_{ij} + \left(\frac{\phi(\beta+m\sigma\phi)}{\sigma F^2}y^ay^l - \frac{2\phi}{F}\underset{al}{\mathcal{S}}(y^a\hat{k}^l)\right)\overset{(a)}{\Gamma}_{ija} + \frac{\sigma}{F}\big(\Lambda^l_{ij} + $$
$$+ M^l_{ij}\big) + (\Lambda_{ija} + M_{ija})\left(\frac{\phi(\beta+m\sigma\phi)}{F^3}y^ay^l - \frac{2\sigma\phi}{F^2}\underset{al}{\mathcal{S}}(y^a\hat{k}^l)\right) \tag{31}$$

where $\Lambda^i_{jl} = \Lambda_{jlk}a^{ik}$ and $M^i_{jl} = M_{jlk}a^{ik}$. In relation (31) it is seen that, besides the $\overset{(a)}{\Gamma}{}^i_{jk} = 0$ terms, the rest express the anisotropic deviation from the Riemannian Christoffel symbols. When $\phi = 0$, i.e. absence of anisotropy, the Finsler Christoffel symbols coincide with the Riemannian ones. From the above relation, for $\overset{(a)}{\Gamma}{}^i_{jk} = 0$ we have $\gamma^i_{jk} \neq 0$. This shows the dependence of γ^i_{jk} from the anisotropy terms.

¿From the relations (2)–(4) we find

$$B^l = \frac{1}{2}\overset{(a)}{\Gamma}{}^l_{jk}y^j y^k + \sigma a^{ml} y^j \partial_{[j}\phi(x)\hat{k}_{m]} \tag{32}$$

$$B^l_k = \frac{\partial G^l}{\partial y^k} = \overset{(a)}{\Gamma}{}^l_{jk}y^j + \sigma a^{ml}\partial_{[k}\phi(x)\hat{k}_{m]} + \frac{1}{\sigma}a^{ml}y^i\partial_{[i}\phi(x)\hat{k}_{m]}y_k \tag{33}$$

$$\begin{aligned} B^l_{kj} = \overset{(a)}{\Gamma}{}^l_{jk} + \frac{1}{2}\Big(& a_{kj}y^r a^{ls}(\partial_{[r}\phi\hat{k}_{s]}) + \tilde{l}_k a^{ls}(\partial_{[j}\phi\hat{k}_{k]}) \\ & + \tilde{l}_j a^{ls}(\partial_{[k}\phi\hat{k}_{s]})\sigma^{-1} - \frac{1}{2}\tilde{l}_i\tilde{l}_j y^k a^{ls}(\partial_{[k}\phi\hat{k}_{s]})\Big) \end{aligned} \tag{34}$$

By substituting in (7) the Berwald type curvature tensor takes the form

$$\begin{aligned} B^l_{hjk} = {} & I^i_{hjk}(x) \\ & + A_{[jk]}\left[\tilde{l}_h a^{im}(\partial_{[j}\phi\hat{k}_{m]})_{;k} + V^m a_{hj}a^{il}(\partial_{[m}\phi\hat{k}_{l]})_{;k} + \tilde{l}_j a^{is}(\partial_{[h}\phi\hat{k}_{s]})_{;k}\right]\sigma^{-1} \\ & - V^m\tilde{l}_h\tilde{l}_j\sigma^{-3}a^{il}(\partial_{[m}\phi\hat{k}_{l]})_{;k} \end{aligned} \tag{35}$$

where

$$\begin{aligned} I^l_{hjk}(x) = R^i_{hjk}(x) + \frac{1}{2}A_{[jk]}\Big[& a^{is}(\partial_{[h}\phi\hat{k}_{s]})(\partial_{[j}\phi\hat{k}_{k]}) + a_{hk}a^{il}a^{ml}(\partial_{[m}\phi\hat{k}_{l]})(\partial_{[j}\phi\hat{k}_{l]}) \\ & - a^{is}(\partial_{[k}\phi\hat{k}_{s]})(\partial_{[h}\phi\hat{k}_{j]})\Big] \end{aligned} \tag{36}$$

By the relation (35), contraction of the indices i and k gives

$$B_{hj} = I_{hj}(x) + \Lambda_{hj}$$

where we put

$$I_{hj} = R_{hj} + \frac{1}{2}\, A_{[ji]} \left[a^{is}(\partial_{[h}\phi\hat{k}_{s]})(\partial_{[j}\phi\hat{k}_{i]}) + \delta^l_h a^{mr}(\partial_{[m}\phi\hat{k}_{r]})(\partial_{[j}\phi\hat{k}_{l]}) - a^{is}(\partial_{[i}\phi\hat{k}_{s]})(\partial_{[h}\phi\hat{k}_{j]}) \right]$$

$$\Lambda_{hj} = A_{[ji]} \left[\tilde{l}_h a^{im}(\partial_{[j}\phi\hat{k}_{m]})_{;i} + V^m a_{hj} a^{il}(\partial_{[m}\phi\hat{k}_{l]})_{;i} + \tilde{l}_j a^{is}(\partial_{[h}\phi\hat{k}_{s]})_{;i} \right] \sigma^{-1} - V^m \tilde{l}_h \tilde{l}_j \sigma^{-3} a^{il}(\partial_{[m}\phi\hat{k}_{l]})_{;i}$$

The term R_{hj} represents the Ricci tensor of Riemannian curvature.

The scalar Riemannian curvature in Finsler spaces for two directions (k, V) one of which (k) represents the axis of anisotropy is given by the form

$$R(x,k,V) = \frac{B_{jihk}(x,k)k^j k^h V^i V^k}{\left(f_{jh}(x,k)f_{ik}(x,k) - f_{ji}(x,k)f_{hk}(x,k)\right) k^j k^h V^i V^k}$$

where the explicit form of B_{jihk} is given by (35).

In the following we consider the curvature for a *tangent Riamannian space* $(M_x : x = \text{constant})$. In a previous work [10] we derived the Ricci tensor of s-curvature of Cartan

$$S_{ih} = -\frac{3(m\sigma^2\phi^2 - \beta^2\phi^2)}{4F^2\sigma^2} a_{ih} - \frac{\phi^2}{4F^2}\hat{k}_i\hat{k}_h + + \frac{\beta\phi^2}{2F^2\sigma^2} \underset{ih}{\mathfrak{S}}(\hat{k}_i y_h) + \frac{3m\sigma^2\phi^2 - 4\beta^2\phi^2}{4F^2\sigma^4} y_i y_h \quad (37)$$

The scalar S of S^i_{jkl} curvature is given by

$$S = \frac{5(\beta^2 - m\sigma^6)\phi^2}{2\sigma F^3} \quad (38)$$

for the indicatrix we get for $F = 1$ and $m = 1$

$$S = \frac{5(\beta^2 - \sigma^2)\phi^2}{2\sigma} \quad (39)$$

In the case where S is independent of the direction V the curvature tensor S^i_{jkl} can be written as

$$S_{ijkl} = S\left(f_{ik}f_{jl} - f_{il}f_{jk}\right) \qquad S : \text{const.} \quad (40)$$

This formula expresses that the curvature parameter S of anisotropy is constant in the fixed point x and for every direction. For example if we have the indicatrix which is of constant curvature hypersurface of Finsler spaces.

In the relation (39) we have $S = 0$ when anisotropy is $\phi(x) = 0$. In this case the microwave background radiation is derived in the framework of a Riemannian space of constant curvature.

The anisotropy parameter S is constant by (39) when $\phi(x) = 0$ and $F = 1$. Physically it means that the norm of $\phi(x)\hat{k}_a$ is constant. Also for this case the direction of V^i with respect to $\hat{k}_a$ should be constant, $V^i_{;k} = 0$.

The indicatrix of M^4_x is a Riemannian space of constant curvature given by [12]

$$R_{\alpha\beta\gamma\delta} = (S+1)(f_{\alpha\gamma}f_{\beta\delta} - f_{\alpha\delta}f_{\beta\gamma}) \tag{41}$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor of the indicatrix.

Remark

If $k^i(x)$ is approximately zero that means by (21) the resultant of spin density tensor is zero. In this case we have also that the Finslerian space-time is independent of the direction of V, that means the space-time is approximately a Riemannian space of constant curvature R.

4 Generalized D'Alembertian

In special relativity the expression of the electromagnetic tensor $F_{\mu\nu}$ in terms of a vector potential A_μ is given by

$$F_{ij} = A_{j,i} - A_{i,j} = A_{j;i} - A_{i;j} \tag{42}$$

Maxwell's equations if the 4-current $J^i = 0$ have the form

$$F^{ij}_{;k} = 0 \tag{43}$$

(electric charged is not present)

The background of these equations is a Riemannian space-time with metric tensor g_{ij}.

If we substitute (42) to (43) we find that the 4-potential satisfies:

$$\Box A_i - g^{ab}A_{a;ib} = 0 \tag{44}$$

where $\Box$ symbolizes the generalized d'Alembertian of Riemannian space-time

$$\Box A_i = g^{ab} A_{a;ib} \tag{45}$$

By the communication rule for covariant differentiation we have

$$A_{a;ib} - A_{a;bi} = R^j_{aib} A_j \tag{46}$$

and hence

$$g^{ab} A_{a;ib} - \left(g^{ab} A_{a;b}\right)_{;i} = -R_{ij} A^j \tag{47}$$

Now, if we impose on A_i the Lorentz condition

$$g^{ab} A_{a;b} = 0 \tag{48}$$

the condition (45) becomes

$$\Box A_i + R_{ij} A^j = 0 \tag{49}$$

The equation is interesting because it brings in the usual wave-operator $\Box$ and the Ricci tensor representing the matter or energy present.

In vacuum we get simply $\Box A_i = 0$ if we neglect the gravitational effect of the electromagnetic field.

In analogy of the Riemannian case we can derive the generalized form of a D'Alembertian in the framework of a Finslerian space-time, in that case it yields

$$\widetilde{F}_{ij} = A_{i;j}(x) - A_{j;i}(x) = \delta_j A_i - \delta_i A_j - L^h_{ij} A_h + L^h_{ji} A_h = F_{ij} \tag{50}$$

where " $|$ " denotes the Cartan covariant derivative in the tangent bundle $T(M)$, where $\dot{\partial}_b A_a(x) = 0$.

Consequently, when the 4-current $J^i = 0$, we get

$$\widetilde{F}^{ik}_{|k} = 0 \tag{51}$$

$$\widetilde{F}^{ij} = A^{j|i} - A^{i|j} \Rightarrow$$
$$\widetilde{F}^{ij}_{|j} = A^{j|i}_{|j} - A^{i|j}_{|j} = f^{mi} A^j_{|m|j} - f^{mj} A^j_{|j|m}$$
$$= f^{mi}\left(A^j_{|j|m} + A^r R^j_{rmj} - A^j_{|r} R^r_{mj}\right) - f^{mj} A^i_{|m|j} = 0$$

$$A^r R^i_r - A^h C^j_{hr} R^{ri}_j - A^{i|j}_{|j} = 0 \tag{52}$$

Where we put $A^{i|j} = f^{jm} A^i_{|m}$ and we used the Lorentz gauge condition $A^i_{|i} = 0$. The tensors R^i_r, R^{ri}_j represent the Ricci tensor and the torsion tensor [5].

The equation (52) represents the generalized D'Alembertain in the Finslerian space-time in the framework of a tangent bundle.

When the gravitational field is absent the generalized wave equation is given by

$$A^{i|j}_{|j} = 0 \qquad \text{or} \qquad f^{jm} A^i_{|mj} = 0 = \Box_f A^i \tag{53}$$

5 Conclusion

The observed anisotropy of the microwave cosmic rediation, represented by a vector $k_a(x)$, can be incorporated in the framework of Finsler geometry. The equations of geodesics are generalized in a Finsler anisotropic space-time. The calculation of a curvature parameter of anisotropy is performed explicitly by the contraction of the S^i_{jkl} curvature. Also, the Maxwell equations are unaffected from the passage to the anisotropic geometry. The Lorentz condition, as well as the generalized D'Alembertian, are shown to be invariant under coordinate transformations. In our case, however, the generalized wave equation includes the anisotropic vector through the L^i_{jk} coefficients and the metric tensor f_{ij}.

Some concluding ideas

- Finsler geometry corresponds to the geometry of space and motion.
- The most important tensors in Finsler geometry are those which reduce to zero in the Riemannian case (e.g. C_{ijk} tensor), or those which have analogues in Riemannian geometry.
- Important points are to consider non-Riemannian Finsler spaces and to know *to what extend* Finsler spaces from Riemannian ones.
- In anisotropic phenomena the geometrical framework of the Riemannian space-time are not sufficient. The most convenient metric geometry is the Finsler geometry.
- The electromagnetic field is incorporated in the Finsler geometry (unified gravitational field) and the paths of charged particles are the geodesics of the space in the contrast to the Riemannian space-time. The equivalence principle under these circumstances can be extended.
- The deviation of geodesics in a Finslerian space-time smoothly extends the concept of the deviation of geodesics of the Riemannian space-time and reveals a physical meaning to Finslerian space-time. Moreover the deviation of geodesics is connected with the study of gravitational waves by considering a Finsler type weak metric analogously with the Riemannian standpoint.

Some useful Finslerian or generalized Finslerian metrics for general relativity and gauge theories.

1. $f_{ij}(x,y) = e^{2\sigma(x,y)} g_{ij}(x)$
2. $f_{\kappa\lambda} = n_{\kappa\lambda} + kB_\kappa B_\lambda$ Beil metric
3. $f_{ij}(x,y) = a_{ij}(x, y(x)) + C_{ijk}y^k + O()$

   Finslerian static gravitational field: $C_{ijk} = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^k}$

4. $f_{ij}(x,y) = \underbrace{(n_{ij} + \varepsilon_{ij}(x))}_{a_{ij}(x)} + h(x,y)$

   Weak Finslerian gravitational field convenient for the study of gravitational waves (traveling tidal forces).

5. $F(x,y) = \sqrt{a_{ij}(x)y^i y^j} + \phi(x)\hat{k}_a y^a$

   Finslerian anisotropic model of space-time.

Metric 1. Has been studied by Tavakol and Van-Den Bergh [13]. They proved that the conditions Ehlers-Pirani-Schild are satisfied and the structure of the universe of the conventional general relativity is preserved if someone uses the metric 1.

Metric 2. Beil metric for the study of unified gauge field theory [14].

Metric 3. Asanov's book for Finslerian static gravitational field [1].

Metric 4-5. P.C. Stavrinos [15], [10].

Open problems

1. Remark-Problem

In a constant Riemannian curvature space we can calculate the quantities $k_a y^a$ from the solution of the deviation of geodesics.

$$\frac{\delta^2 k^\beta}{\delta u^2} - K(k_a y^a) y^\beta = 0 \qquad y^a = \frac{dx^a}{dt} \tag{54}$$

Solutions:

$$k_a y^a = \frac{1}{2} u^2 KCD + A_u + B \qquad K, A, B, C, D : \text{const.} \tag{55}$$

We can enter the terms from (55) in the anisotropic Finslerian space-time of metric

$$F(x,y) = \left(a_{ij}(x) y^i y^j\right)^{1/2} + \phi(x) \hat{k}_a y^a \tag{56}$$

where $\phi(x)\hat{k}_a = k_a$.

If we get special coordinates in the pseudo-Riemannian space of constant curvature and we also get a space of constant curvature Berwald type

$$H_{ijkl} = K\left(f_{ik} f_{jl} - f_{il} f_{jk}\right)$$

we ask what type equations we can get in order to explain the anisotropic part of the background microwave radiation in the framework of a Finslerian (anisotropic) space-time of constant curvature? The analogous standpoint valids in the Riemannian space when the Riemann curvature is constant. It is connected with the equations of the background radiation of an isotropic universe.

2. In Finslerian approach the curvature of Finsler space-time is characterized by the tensors $F^i_{jkl}, P^i_{jkl}, K^i_{jkl}$. (In the Riemannian space there is only one R^i_{jkl}). Thus, the question arises when it is possible to find a full interpretation of the curvature of a Finsler space in terms of geodesics deviations.

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