

Randers Metrics and Their Curvature Properties

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Abstract

In this paper, we give some historical remarks on the Randers metrics and introduce the important roles of Randers metrics in studying the Finsler metrics of constant curvature. At the same time, we discuss some special curvature properties of Randers metrics, including the mean Berwald curvature, the mean Landsberg curvature and the mean Cartan torsion.

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1 Historical Remarks on The Randers Metrics

Randers metrics are among the simplest Finsler metrics. They are expressed in the form

$$F = \alpha + \beta,$$

where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on a differentiable manifold M and $\beta := b_i(x)y^i$ is a 1-form on M with

$$\|\beta\|_\alpha(x) := \sup_{y \in T_x M} |\beta(y)| / \alpha(y) < 1$$

for any point $x \in M$. Randers metrics were first introduced by physicist G. Randers[R] in 1941 from the standpoint of general relativity. Later on, these metrics were applied to the theory of the electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics. Up to now, many Finslerian geometers have made a great of efforts in investigation on the geometric properties of Randers metrics.

- 1974, M. Matsumoto: $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel with respect to α , namely,

$$b_{i|j} := \frac{\partial b_i}{\partial x^j} - b_k \gamma_{ij}^k = 0, \quad (1)$$

where γ_{ij}^k denote the Christoffel symbols of α .

- 1979, S. Kikuchi[Ki]:
 - A. a Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel with respect to α .
 - B. a Randers metric $F = \alpha + \beta$ is locally Minkowskian if and only if α is flat and β is parallel with respect to α .

- 1997, S. Bácsó and M. Matsumoto[BaMa]: a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is a close 1-form. We note that, for a Randers metric $F = \alpha + \beta$, β is close if and only if $F = \alpha + \beta$ is pointwise projective to α . From this, we can easily see that, a Randers metric $F = \alpha + \beta$ is a projective flat Finsler metric if and only if β is close and α is locally projectively flat.

One of the fundamental problem in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature (the flag curvature in Finsler geometry is an analogue of the sectional curvature in Riemannian geometry). Of course, Finslerian geometers are very concerned with the conditions that a Randers metric is of constant flag curvature.

- 1977, H. Yasuda and H. Shimada[YS]: a Randers metric $F = \alpha + \beta$ is of constant flag curvature $\lambda > 0$ if and only if β is a Killing 1-form of constant length with respect to α with additional conditions.
- 2001, Z. Shen[Sh1]: considered the conditions that a Randers metric is of constant flag curvature again and gave a condition that a Randers metric is of constant Ricci curvature.
- 2001, D. Bao and C. Robles[BaRo]: gave an equivalent condition for Randers metrics to be of constant curvature.
- An interesting problem is: under what conditions, must a Finsler metric with flag curvature $K = 0$ be locally Minkowskian? H. Akber-Zadeh[AZ] proved that any positively complete Finsler metric with $K = 0$ must be locally Minkowskian if its first and

second Cartan torsions are bounded. Fortunately, for Randers metrics, the first and second Cartan torsions satisfy

$$\|C\| < \frac{3}{\sqrt{2}}, \quad \|\tilde{C}\| < \frac{27}{2}$$

respectively[BaChSh][Sh2][Sh3]. Hence, if $F = \alpha + \beta$ is a positively complete Randers metric, then $K = 0$ if and only if F is locally Minkowskian[Sh3].

- 2001, Z. Shen[Sh4]: classification of locally projectively flat Randers metrics with constant flag curvature.

Theorem 1.1. Let $F = \alpha + \beta$ be an n -dimensional Randers metric with constant curvature $K = \lambda$. Suppose that F is locally projectively flat. Then F is either locally Minkowskian ($\lambda = 0$) or after a scaling, isometric to a Finsler metric on the unit ball B^n in the following form

$$F_a = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2} \pm \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad (2)$$

$$y \in T_x \mathbf{R}^n,$$

where $a \in \mathbf{R}^n$ is a constant vector with $|a| < 1$. The Randers metric in (2) has the following properties: (a) $K = -1/4$; (b) $S = \pm \frac{1}{2}(n+1)F_a$; (c) $E = \pm \frac{1}{4}(n+1)F_a^{-1}h$; (d) $J \pm \frac{1}{2}F_a I = 0$ and (e) all geodesics of F_a are straight lines, where h denotes the angular metric of F_a and S, E, J and I denote the S -curvature, the mean Berwald curvature, the mean Landsberg curvature and the mean Cartan torsion, respectively.

Remark 1.1 By Theorem 1.1, every locally projectively flat Randers metric with $K = 0$ must be locally Minkow-

skian.

Example 1.1([Z. Shen,2001,[Sh7]) Let

$$F(x, y) =$$

$$\frac{\sqrt{(1 + \langle a, x \rangle)^2 |y|^2 - 2(1 + \langle a, x \rangle) \langle a, y \rangle \langle x, y \rangle + \langle a, y \rangle^2 |x|^2}}{(1 + \langle a, x \rangle)^2} + \frac{(1 + \langle a, x \rangle) \langle b, y \rangle - \langle a, y \rangle \langle b, x \rangle}{(1 + \langle a, x \rangle)^2}, \quad y \in T_x \mathbf{R}^n,$$

where $a, b \in \mathbf{R}^n$ and $|b| < 1$. Then F is a projective Randers metric with $K = 0$. Hence, F is a locally Minkowski metric.

2 The important roles of Randers metrics in studying the Finsler metrics of constant curvature

Beltrami shows that a Riemannian metric is locally projectively flat if and only if it is of constant curvature. A natural question arises: for a Finsler metric, is this still true?

All of the known non-Riemannian Finsler metrics with constant curvature:

Part I. Projectively flat Finsler metrics

- Hilbert-Klein metric on a strongly convex domain $\Omega \subset R^n$: complete, reversible, $K = -1$.
- Funk metric on a strongly convex domain $\Omega \subset R^n$: positively complete, non-reversible, $K = -1/4$.
- Bryant metrics(1995) on S^2 or S^n : non-reversible, $K = 1$.

- Shen metric(2001,[Sh7]) on $B^n \subset R^n$: non-reversible, $K = 0$.

Part II. Non-projectively flat Finsler metrics

- Bao-Shen metrics(2000, [BaSh]): the first family of non-projectively flat Finsler metrics on S^3 with $K = 1$. Their examples are in the form $F = \alpha + \beta$, which just are Randers metrics.
- Shen's metrics(2001, [Sh3][Sh5]): non-projectively flat Finsler metrics with constant curvature $K = -1, 0, 1$, respectively. All of Shen's these metrics are still in the form $F = \alpha + \beta$. Shen's main method is so-called the shortest time problem[Sh3], a mathematical model characterized by Randers metrics.

So, we may say that Randers metrics play an very important role in knowing the geometric structures and properties of Finsler metrics with constant curvature.

The following are the non-projectively flat Randers metrics with $K = -1, 0, 1$ respectively, constructed by the second author. See [Sh3][Sh5] for more details.

We know that Minkowski spaces are Finsler spaces with $K = 0$. According to [AZ], any positively complete Finsler metric with $K = 0$ must be locally Minkowskian if its first and second Cartan torsions are bounded. A natural question is: are there any positively complete non-Minkowskian Finsler metrics with $K = 0$? This problem remains open. Firstly, we give the following Theorem 2.1.([Sh3]) Let $n \geq 2$ and

$$\Omega := \{p = (x, y, \bar{p}) \in R^2 \times R^{n-2} | x^2 + y^2 < 1\}.$$

Define

$$F(\mathbf{y}) := \frac{\sqrt{(-yu + xv)^2 + |\mathbf{y}|^2(1 - x^2 - y^2)} - (-yu + xv)}{1 - x^2 - y^2}, \quad (3)$$

where $\mathbf{y} = (u, v, \bar{\mathbf{y}}) \in T_p\Omega = R^n$ and $p = (x, y, \bar{p}) \in \Omega$. Then F is a Finsler metric on Ω satisfying $K = 0$, $\mathbf{S} = 0$.

A easy computation shows that the Randers metric in (3) is not Douglas metric, hence not locally projectively flat. Therefore, it is not locally Minkowskian. Because Randers metrics have bounded first and second Cartan torsions, the Randers metric in (3) is not positively complete.

Now, let us describe our another example. Let \langle, \rangle denote the standard Riemannian metric on S^2 and \mathbf{x} denote the vector field on S^2 defined by

$$\mathbf{x}_p := (-y, x, 0) \text{ at } p = (x, y, z) \in S^2.$$

Let $F := \alpha + \beta$, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha := \frac{\sqrt{\epsilon^2 \langle \mathbf{x}, \mathbf{y} \rangle^2 + \langle \mathbf{y}, \mathbf{y} \rangle (1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle)}}{1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle}, \quad (4)$$

$$\beta := -\frac{\epsilon \langle \mathbf{x}, \mathbf{y} \rangle}{1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle},$$

where ϵ is an arbitrary number. F is defined on the whole sphere for $|\epsilon| < 1$ and it is defined only on the open disks around the north pole and south pole with radius $\rho = \sin^{-1}(1/|\epsilon|)$ for $|\epsilon| \geq 1$. Note that when $\epsilon = 0$, F is the standard Riemannian metric on S^2 .

Theorem 2.2([Sh5]) Let $F = \alpha + \beta$ be the Finsler metric on S^2 defined in (4). It has the following properties

- (i) $K = 1$;
 - (ii) $S = 0$;
 - (iii) F is not locally projectively flat unless $\epsilon = 0$;
 - (iv) the Gauss curvature \bar{K} of α is not a constant unless $\epsilon = 0, \pm 1$. When $\epsilon = 0$, $\bar{K} = 1$; when $\epsilon = \pm 1$, $\bar{K} = -4$
- Remark 2.1** We know that when a Randers metric $F = \alpha + \beta$ is locally projectively flat, α is of constant curvature. However, for the Finsler defined in (4), when $\epsilon = \pm 1$, $\bar{K} = -4$ and F is not locally projectively flat.

Similarly, let \langle, \rangle denote the standard Klein metric on the unit disk \mathbf{D}^2 and \mathbf{x} denote the vector field on \mathbf{D}^2 defined by

$$\mathbf{x}_p = (-y, x) \text{ at } p = (x, y) \in \mathbf{D}^2.$$

For an arbitrary number ϵ , let $F := \alpha + \beta$, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha := \frac{\sqrt{\epsilon^2 \langle \mathbf{x}, \mathbf{y} \rangle^2 + \langle \mathbf{y}, \mathbf{y} \rangle (1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle)}}{1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle}, \quad (5)$$

$$\beta := -\frac{\epsilon \langle \mathbf{x}, \mathbf{y} \rangle}{1 - \epsilon^2 \langle \mathbf{x}, \mathbf{x} \rangle}.$$

F is a Finsler metric defined on the disk $\mathbf{D}^2(\rho)$ with radius $\rho = 1/\sqrt{1 + \epsilon^2}$. Note that when $\epsilon = 0$, F is the Klein metric on the unit disk.

Theorem 2.3([Sh5]) Let $F = \alpha + \beta$ be a Finsler metric on the disk $\mathbf{D}^2(\rho)$ defined in (5). It has the following properties:

- (i) $K = -1$;
- (ii) $S = 0$;
- (iii) F is not locally projectively flat if $\epsilon \neq 0$;
- (iv) the Gauss curvature \bar{K} of α is not constant unless $\epsilon = 0$. When $\epsilon = 0$, $\bar{K} = -1$.

Remark 2.2 Although the Randers metrics defined in (5) is of negative constant curvature, the Riemannian metric α is not of negative constant curvature when $\epsilon \neq 0$.

3 Some special curvature properties of Randers metrics

The Funk metric on a strongly convex domain Ω in \mathbf{R}^n has many special curvature properties: (a) $K = -1/4$; (b) $\mathbf{S} = \pm \frac{1}{2}(n+1)F$; (c) $\mathbf{E} = \pm \frac{1}{4}(n+1)F^{-1}h$; (d) $\mathbf{J} \pm \frac{1}{2}F\mathbf{I} = 0$ and (e) all geodesics of F are straight lines. When Ω is just the unit ball \mathbf{B}^n in \mathbf{R}^n , the Funk metric

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2}, \quad y \in T_x \mathbf{R}^n. \quad (6)$$

The Funk metric in (6) is a special Randers metric. Motivated by the properties of Funk metrics, we want to see whether Randers metrics have the properties same as that of Funk metrics.

Firstly, we briefly recall the definitions of the mean Berwald curvature, the mean Cartan torsion and the mean Landsberg curvature. For a non-zero vector $y \in T_p M$, the mean Berwald curvature $\mathbf{E}_y = E_{ij}(x, y)dx^i \otimes dx^j : T_p M \times T_p M \rightarrow R$ is defined by

$$E_{ij} := \frac{1}{2} \frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j}(x, y). \quad (7)$$

The mean Cartan torsion $\mathbf{I}_y = I_i(x, y)dx^i : T_p M \rightarrow R$ is defined by

$$I_i := \frac{1}{4} g^{jk} [F^2]_{y^i y^j y^k} = \frac{\partial}{\partial y^i} \left[\ln \sqrt{\det(g_{jk})} \right]. \quad (8)$$

$$\iff I_i = g^{jk} C_{ijk}.$$

Further, for a non-zero vector $y \in T_p M$, the mean Landsberg curvature $\mathbf{J}_y = J_i(x, y) dx^i : T_p M \rightarrow R$ is defined by

$$J_i := -\frac{1}{2} F F_{y^l} g^{jk} \frac{\partial^3 G^l}{\partial y^i \partial y^j \partial^k}. \quad (9)$$

$$\iff J_i = g^{jk} L_{ijk}.$$

Now, let $F = \alpha + \beta$ be a Randers metric on a manifold M with $\|\beta\|_\alpha(x) := \sup_{y \in T_x M} |\beta(y)|/\alpha(y) < 1$. An easy computation yields

$$g_{ij} := \left[\frac{F^2}{2} \right]_{y^i y^j} = \frac{F}{\alpha} \left(a_{ij} - \frac{y_i y_j}{\alpha} \right) + \left(\frac{y_i}{\alpha} + b_i \right) \left(\frac{y_j}{\alpha} + b_j \right), \quad (10)$$

where $y_i := a_{ij} y^j$. By an elementary argument in linear algebra, we obtain

$$\det(g_{ij}) = \left(\frac{F}{\alpha} \right)^{n+1} \det(a_{ij}). \quad (11)$$

Define $b_{i|j}$ by

$$b_{i|j} \theta^j := db_i - b_j \theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \bar{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of α . Let

$$r_{ij} := \frac{1}{2} (b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2} (b_{i|j} - b_{j|i}),$$

$$s^i_j := a^{ih} s_{hj}, \quad s_j := b_i s^i_j, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then the geodesic coefficients G^i are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2F} y^i - s_0 y^i + \alpha s^i_0, \quad (12)$$

where \bar{G}^i denote the geodesic coefficients of α , $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$ and $s^i{}_o := s^i{}_jy^j$.

For a Randers metric, the S -curvature is given by

$$\mathbf{S} := (n+1) \left\{ \frac{e_{00}}{2F} - s_0 - d\rho(y) \right\}, \quad (13)$$

where $\rho := \ln \sqrt{1 - \|\beta\|_\alpha^2(x)}$ [ChSh][Sh3][Sh6]. It is easy to see

$$E_{ij} = \frac{1}{2} \mathbf{S}_{y^iy^j}.$$

We have the following

Lemma 3.1 Let $F = \alpha + \beta$ be a Randers metric on a manifold M . For a scalar function $c = c(x)$ on M , the following are equivalent:

- (a) $\mathbf{S} = (n+1)cF$;
- (b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. From (13), we see that $\mathbf{S} = (n+1)cF$ if and only if

$$e_{ij} = (s_i + \rho_i)b_j + (s_j + \rho_j)b_i + 2c(a_{ij} + b_ib_j), \quad (14)$$

$$s_i + \rho_i + 2cb_i = 0. \quad (15)$$

Similarly, $e_{00} = 2c(\alpha^2 - \beta^2)$ is equivalent to the following identity

$$e_{ij} = 2c(a_{ij} - b_ib_j). \quad (16)$$

From these, we can prove the equivalence. **Q. E. D.**

On the other hand, we have the following

Lemma 3.2 Let $F = \alpha + \beta$ be a Randers metric on a manifold M . For a scalar function $c = c(x)$ on M , the following are equivalent

- (a) $\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h$;
- (b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. From (13) and $E_{ij} = (1/2)\mathbf{S}_{y^iy^j}$, we have

$$E_{ij} = \frac{n+1}{4} \left[\frac{e_{00}}{F} \right]_{y^iy^j}. \quad (17)$$

Notice that $h_{ij} := g_{ij} - F_{y^i}F_{y^j} = FF_{y^iy^j}$. So $\mathbf{E} = \frac{n+1}{2}cF^{-1}h$ if and only if

$$E_{ij} = \frac{n+1}{2}cF_{y^iy^j}. \quad (18)$$

From (17),(18), we can prove the equivalence. **Q. E. D.**

By Lemma 3.1 and Lemma 3.2, we have the following Theorem 3.1 Let $F = \alpha + \beta$ be a Randers metric on a manifold M . For a scalar function $c = c(x)$ on M , the following are equivalent:

- (a) $\mathbf{S} = (n+1)cF$;
- (b) $\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h$.

Remark 3.1 For a Finsler metric F on an n -dimensional manifold M , we say that F has constant E -curvature c if $\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h$. Similarly, we say that F has constant S -curvature c if $\mathbf{S} = (n+1)cF$. Theorem 3.1 shows that, for a Randers metric F , it has constant S -curvature c if and only if it has constant E -curvature c . However, for a general Finsler metric, this conclusion is not true. Recently, the second author has found an example with the properties $\mathbf{E} = \text{constant}$ but $\mathbf{S} \neq \text{constant}$.

Now, let us consider the mean Landsberg curvature \mathbf{J} . We have the following

Lemma 3.3 For a Randers metric $F = \alpha + \beta$, the mean Cartan torsion $\mathbf{I} = I_idx^i$ and the mean Landsberg curvature $\mathbf{J} = J_idx^i$ are given by

$$I_i = \frac{1}{2}(n+1)F^{-1}\alpha^{-2}\{\alpha^2b_i - \beta y_i\}, \quad (19)$$

$$J_i = \frac{n+1}{4}F^{-2}\alpha^{-2}\{2\alpha[(e_{i0}\alpha^2 - y_ie_{00}) - 2\beta(s_i\alpha^2 - y_is_0) + s_{i0}(\alpha^2 + \beta^2)] + \alpha^2(e_{i0}\beta - b_ie_{00}) + \beta(e_{i0}\alpha^2 - y_ie_{00}) - 2(s_i\alpha^2 - y_is_0)(\alpha^2 + \beta^2) + 4s_{i0}\alpha^2\beta\}. \quad (20)$$

From Lemma 3.3, we can prove the following
Lemma 3.4 Let $F = \alpha + \beta$ be a Randers metric on a manifold M . For a scalar function $c = c(x)$ on M , the following are equivalent

- (a) $\mathbf{J} + cF\mathbf{I} = 0$;
- (b) $e_{00} = 2c(\alpha^2 - \beta^2)$ and β is close.

By Lemma 3.1 and Lemma 3.4, we immediately get the following

Theorem 3.2 Let $F = \alpha + \beta$ be a Randers metric on a manifold M . For a scalar function $c = c(x)$ on M , the following are equivalent

- (a) $\mathbf{J} + cF\mathbf{I} = 0$;
- (b) $\mathbf{S} = (n + 1)cF$ and β is closed.

There are lots of Randers metrics satisfying $\mathbf{S} = (n + 1)cF$ or $\mathbf{J} + cF\mathbf{I} = 0$. Besides the Randers metric in (6) and the Randers metrics in (2) with $c = 1/2$, we give the following

Example 3.1 For an arbitrary number ϵ with $0 < \epsilon \leq 1$, define

$$\alpha := \frac{\sqrt{(1 - \epsilon^2)(xu + yv)^2 + \epsilon(u^2 + v^2)(1 + \epsilon(x^2 + y^2))}}{1 + \epsilon(x^2 + y^2)}$$

$$\beta := \frac{\sqrt{1 - \epsilon^2}(xu + yv)}{1 + \epsilon(x^2 + y^2)}.$$

We have

$$\|\beta\|_\alpha = \sqrt{1 - \epsilon^2} \sqrt{\frac{x^2 + y^2}{\epsilon^2 + x^2 + y^2}} < 1.$$

Thus $F = \alpha + \beta$ is a Randers metric. By a direct computation, we obtain

$$e_{00} = 2c(\alpha^2 - \beta^2),$$

where

$$c = \frac{\sqrt{1 - \epsilon^2}}{2(\epsilon + x^2 + y^2)}.$$

Moreover,

$$\mathbf{J} + cF\mathbf{I} = 0.$$

But F does not have constant curvature.

We note that, in the properties of Randers metrics in (2) and (6), the most fundamental properties are (a) and (d). In fact, for the Randers metrics of constant curvature which satisfy $\mathbf{J} + cF\mathbf{I} = 0$, we have the following classification theorem, that is

Theorem 3.3 Let $F = \alpha + \beta$ be a Randers metric of constant curvature $K = \lambda$ on a manifold M . Suppose that F satisfies $\mathbf{J} + cF\mathbf{I} = 0$ for some scalar function $c(x)$ on M . Then $\lambda = -c^2 \leq 0$. F is either locally Minkowskian ($\lambda = -c^2 = 0$) or in the form (2) ($\lambda = -c^2 = -1/4$) after a scaling.

Proof. By Lemma 3.4 and the assumption $\mathbf{J} + cF\mathbf{I} = 0$, we know that

$$e_{00} = 2c(\alpha^2 - \beta^2) \text{ and } \beta \text{ is closed.}$$

Hence the Douglas curvature D of F vanishes. On the other hand, by the assumption $K = \lambda$, we know that the Weyl curvature W vanishes. Therefore, F is locally projectively flat. So the theorem follows from Theorem 1.1. Q. E. D.

4 A generalization of Theorem 3.3

Let (M, F) be a Finsler manifold of dimension n . Fix a local frame $\{e_i\}$ for TM . Let

$$\tau(x, y) := \ln \left[\frac{\sqrt{\det(g_{ij}(x, y))}}{\text{Vol}(\mathbf{B}^n(1))} \cdot \text{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i e_i) < 1\} \right].$$

We call τ the *distorsion*[Sh6].

Theorem 4.1[ChMoSh] Let F be an isotropic Finsler metric on a manifold M of dimension n . Suppose that the mean Landsberg curvature \mathbf{J} satisfies $\mathbf{J} + cFI = 0$ for some scalar function $c(x)$ on M . Then the flag curvature K and the distorsion τ satisfy

$$\frac{n+1}{3}K_{.l} + \left(K + c^2 - \frac{c_{x^m}y^m}{F} \right) \tau_{.l} = 0. \quad (21)$$

Further,

(a) If $c = \text{constant}$, then $(K + c^2)^{\frac{n+1}{3}} e^\tau$ is independent of $y \in T_x M$.

(b) If $K = K(x)$ is independent of $y \in T_x M$ at some point $x \in M$, then either F_x is Euclidean or $c = \text{constant}$ and $K(x) = -c^2 \leq 0$.

From Theorem 4.1, we see that, for any Finsler metric F with flag curvature $K = K(x)$ depending only on $x \in M$, if it satisfies $\mathbf{J} + cFI = 0$, then F is either Riemannian or $K = -c^2$ is a nonpositive constant. Obviously, for a Randers metric of constant curvature satisfying $\mathbf{J} + cFI = 0$ for some $c(x)$ on M , we have $K = \text{constant} = -c^2 \leq 0$. Further, we have the following

Theorem 4.2 Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n . Suppose that (1) flag curvature $K = K(x)$ is independent of $y \in T_x M$; (2) $\mathbf{J} + c(x)FI = 0$ ($\iff S = (n+1)c(x)F$ and β is closed).

Then F is locally projectively flat and $K = \text{constant} = -c^2 \leq 0$.

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OPEN PROBLEMS

1. Complete to classify Randers metrics with constant curvature.
2. Complete to classify projectively flat (α, β) -metrics with constant curvature.

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